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The Power of Incorrect Answers

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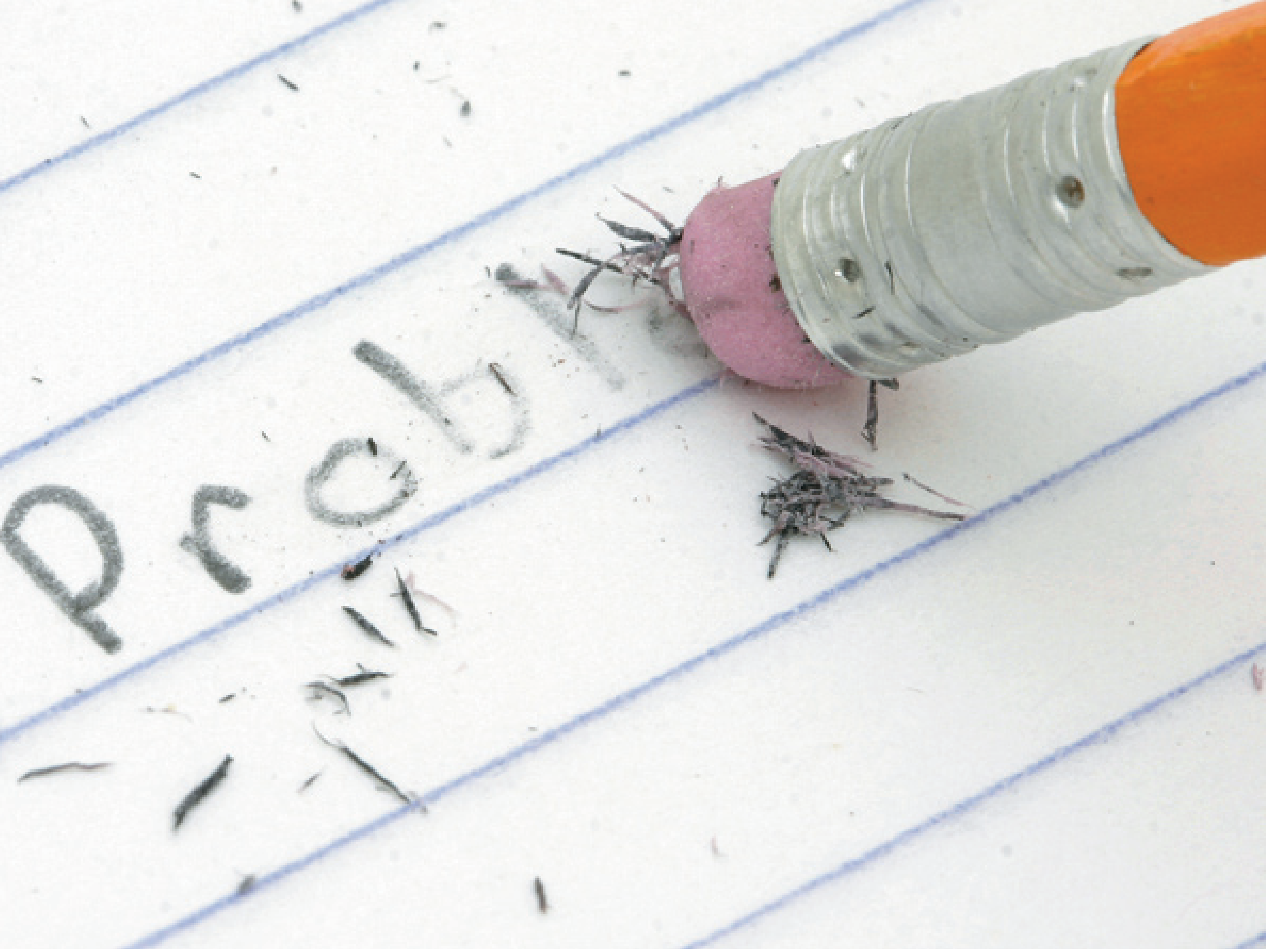
To bolster students' ability to prove as well as develop mathematical argumentation skills, create an environment in which students must regularly explain and justify their thinking.

Mathematical argumentation is an important skill. It leads to the process of proof, which is one area that students are being asked to master by the end of secondary school. Encouraging explanation and justification in math class allows this skill to develop. Explaining and justifying their ideas forces students to think deeply about mathematics and informally prove in preparation for more formal proof writing. However, developing a classroom in which explanation and justification are common is a challenge.

The process of proof cannot simply begin in the later years of secondary education. Students must develop the skills required to reason mathematically from kindergarten through grade 12. Our goal with this article is to help facilitate this process by putting



into practice a tool from mathematics education research. We share a helpful framework for reflecting on mathematics discourse in the classroom. We also analyze actual classroom episodes using this framework and think about how to apply the framework in practice.

NCTM emphasizes the importance of proof. "By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments" (2000, p. 56). Beginning the process of mathematical argumentation in middle school will help students with proof when they reach high school.

THE MATH TALK FRAMEWORK

We use the Math Talk Learning Community Framework (Hufferd-Ackles, Fuson, and Sherin 2004), herein referred to as the Math Talk Framework (see **table 1**), as a way to assess discourse and develop the processes of explanation and justification. It assesses discourse through the following four components:

1. Questioning
2. Explaining mathematical thinking
3. Noting the source of mathematical ideas
4. Taking responsibility for learning

We focus most prominently on the component of explaining mathematical thinking; however, because each component is inherently connected to the others, we also include an examination of the other three.

The Math Talk Framework is broken into four levels, ranging from 0 to 3. Level 0 represents a more teacher-centered classroom; in a level 3 classroom, teacher and students share responsibility for student learning through discourse. The progression within each component from level 0 to 3 requires a shifting of the focus from the teacher to the students—the classroom community—so that students become primarily responsible for questioning, explaining, and providing mathematical ideas. To promote explanation and justification of ideas and strategies, a teacher should work to progress from a lower level to a higher level.

At levels 2 and 3, the focus is no longer on the teacher but on the classroom community in which the teacher and students play essential roles. With the teacher serving a more peripheral role, students are responsible for explaining and justifying their ideas and strategies to the other members of the learning community. By doing so, students can begin to build on one another's mathematical thinking, which provides a rich context for discourse and learning (Sherin, Louis, and Mendez 2000). In addition, Manouchehri and St. Jon (2006) argue that "Students must become able to make and state mathematical observations on their own, take ownership of the thinking that must be done, and break away from the belief, fostered by much of the schooling process, that authority resides only in books and teachers" (p. 550). Learning communities at levels 2 and 3 help to accomplish this because, without the teacher to ascribe "correctness," students must evaluate each other's ideas and strategies, forcing them to explain and justify until they reach a correct answer and full understanding of the concept.

Table 1 This Math Talk Framework helps guide teachers when reflecting on mathematics discourse in the classroom.

Level	Classroom Characteristics
0	The classroom is teacher-centered, with the teacher serving as the primary questioner, source of mathematical ideas, and authority to verify the correctness of solutions. Students are mostly passive, giving only short answers as prompted by the teacher.
1	The teacher begins to share responsibility for learning with the students. While maintaining the central role, the teacher begins to probe student thinking. Students' explanations are brief.
2	The teacher and students share responsibility for learning. The teacher probes students more deeply and uses student errors for learning opportunities. Students ask questions of one another, explain and defend their ideas, and build on the ideas of their peers.
3	The teacher serves a more peripheral role. Without prompting from the teacher, students question one another, explain and justify their ideas, and work together to compare and contrast strategies and solutions.

Adapted from Hufferd-Ackles, Fuson, and Sherin (2004)

Fig. 1 Excerpt 1: Conversation demonstrates how Mr. D. places the responsibility for determining the solution on his students.

Mr. D.: Well, what if I do this? You have $2b \times 3a$. What would that be equal to? [Mr. D. writes $(2b)(3a)$ on the board.]
Rick: $2b \times 3a$
Mr. D.: $2b \times 3a$? So you're saying it stays the same?
Rick: Yeah.
Mr. D.: Why?
Rick: Because there aren't any like terms.
Mr. D.: There aren't any like terms? Okay. What makes you say they're not like terms?
Rick: Uh, the variables aren't the same?
Mr. D.: Oh, the variables aren't the same.
Rick: Yes.
Mr. D.: So, because they're not like terms . . . . [Voice trails]
Rick: We can't multiply them together.
Mr. D.: We can't multiply them together. Okay. Interesting thought. Does anybody disagree with him?
[Long pause; no students respond.]
Mr. D.: No? Well, let's look at this. That's an interesting comment, Rick. I like that. Let's build on that.

RESEARCH INTO PRACTICE

Episode Illustrating a Level 2/3

To see how the Math Talk Framework can be applied within a math class, we focus on the efforts of Mr. D., an eighth-grade algebra teacher. We trace the development of a learning community in his classroom through the use of three excerpts. The first two excerpts come from the same lesson on simplifying variable expressions; the third excerpt comes from a lesson on order of operations.

Mr. D. began the lesson with the following example:

$$\text{Simplify } (2b)(-3).$$

After the class decided that the answer was $-6b$, Mr. D. moved to a second example of simplifying $(2b)(3a)$. The verbatim dialogue, transcribed in **figure 1**, demonstrates how Mr. D. places the responsibility for determining the solution on his students.

Mr. D. resisted the temptation to tell Rick that he was wrong. Instead, he allowed Rick to explain his thinking and reasoning, albeit wrong, about the process. In so doing, he uncovered student misconceptions. Additionally, by responding to Rick's answer as he did, Mr. D. validated Rick's contribution to the conversation before bringing it to the community for evaluation. Cooke and Adams explained, "In order to overcome student fears that their answers will be wrong or that other students will not listen respectfully, the teacher must work to build trust" (1998, p. 35). By confirming that he appreciated Rick's idea for its value in discussion, Mr. D. was building trust in this learning community.

The second excerpt, transcribed in **figure 2**, immediately followed the lesson. Mr. D. gave the community the opportunity to have a mathematical argument about the difference between adding and multiplying variable expressions that were raised from Rick's idea about like terms.

Mr. D.'s decision to focus the class discussion on Rick's idea provided a rich context for mathematical argumentation. For example, Mike summarized that all operations can be carried out when there is only one variable. However, he said you cannot add two different variables together. Ali then built on this idea by explaining that two different variables can be combined through multiplication because you can "split [the terms] apart" and then multiply. Although Mr. D. might have taken the opportunity to suggest that Ali use a different word than "split," which could suggest addition, these ideas might not have emerged if Mr. D. had simply identified Rick's idea as "wrong" and told students that like terms are only required for addition and subtraction. The discourse that transpired allowed students to think about the differences among adding, subtracting, multiplying, and dividing variable expressions and which operations require like terms.

When managing group discussion, these questions always arise:

1. How much do individuals in the classroom gain?

Fig. 2 Excerpt 2: In this discussion, the class considers the information elicited from Rick about like terms.

Mr. D.: Up here though. [Motions to (2*b*)(–3)] Are these two like terms? Because we carried out the multiplication? Leah?
Leah: Um, when you have the variables [inaudible] because, like, you can multiply. Because you're multiplying 2×3 anyway.
Mr. D.: Mmmmm.
Leah: So, 2×3 is 6. And then, you can't, and then, it's gonna be negative. It's gonna be negative, $-6 \times b$. But, see, then you multiply $2b \times 3$ and it becomes $3a$. Okay. No, you can't do the first one the way we did it.
Mr. D.: Okay. So now you're saying no. Mike?
Mike: You can add, subtract, divide, and multiply the amount of variable you have. But you can't add another variable with it.
Mr. D.: Okay. Let's try that again. Say that again. Repeat it.
Mike: You can't add. You can add, subtract, multiply, or divide a variable. But you cannot add another variable to it.
Mr. D.: Another variable to it?
Mike: Like, a different variable.
Mr. D.: Okay. Okay. So we're adding. You cannot add two different types of variables together? Right?
Mike: Yeah.
Mr. D.: Okay. What about multiplying two different variables together?
[He pauses for students to consider.]
Mr. D.: Ali?
Ali: For that, for the second problem [(2*b*)(3*a*)], isn't it basically $2 \times b \times 3 \times a$? So, like, if you split the 2 and the *b* and the 3 and the *a* apart, it would be the same problem, right?
Mr. D.: Right.
Ali: So you could multiply 2×3 to give you 6. Right. It would be $6ab$.
Mr. D.: Okay. So, you're saying the answer to this is $6ab$?
Ali: Exactly.
Mr. D.: Kaylen?
Kaylen: Oh yeah. There's, there's always like, when I think of algebra when my brother was doing it. He would always do something like, you know, $a \times b = ab$. But, like, you can't do that with addition or subtraction, because you need like terms. But I was thinking that multiplying and dividing the terms don't matter.

2. Are silent members engaged in the discussion?

In excerpt 1 (**fig. 1**), Mike, Ali, and Kaylen were silent observers. Because these three students joined the conversation in excerpt 2 (**fig. 2**), it is clear that they felt compelled to contribute their ideas after actively listening and considering the mathematics. Their silent participation became vocal participation. Hillen and Smith (2007) proposed that when the discourse is rich and full of explanation and justification, silent members of a learning community can gain new understanding.

Because Mr. D., rather than the students, asked most of the questions in these two excerpts, the interaction did not reach level 3 of the Math Talk Framework. In these excerpts, Mr. D. probed his students to explain their ideas and justify them as they argued for their point. By maintaining part of his role as questioner, Mr. D. steered students in a direction to facilitate understanding while allowing them to

Reflect and Discuss

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we can identify and explore our own practices and underlying beliefs.

The following questions are prompts to help you reflect on the article and on how the authors' ideas might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as to discuss it with colleagues.

To achieve a level 2 or 3 on the Math Talk Framework, the teacher needs to take a more peripheral role and allow students to generate the questions and discussion.

- What specific strategies might you employ in your classroom to help develop this culture?
- In what ways can you encourage your students to ask more questions of one another so that you are not the sole facilitator of the classroom discourse?

In the first excerpt (see **fig. 1**), Rick struggled with the notion of "like terms" and what operations one must consider "like terms."

- How do you help students make sense of like terms when using variables?
- How is this similar to or different from students' fragile understanding of the need for common denominators in certain situations?

In the second excerpt (see **fig. 2**), several new students enter the conversation who were silent earlier. Research mentioned in the article suggests that silent participants can benefit from discourse that is rich and full of justification.

- What are some strategies to assess whether these silent participants really gain new understanding from that day's discussion?
- What techniques do you use to provide closure to any discussion to ensure that all students are taking away new understanding?

The authors provided two alternatives for excerpt 3 that Mr. D. might have used to engage the students in more fruitful discourse.

- What other ideas might you have for altering his response to achieve a level 2 or 3?

You are invited to tell us how you used "Reflect and Discuss" as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Send letters to *Mathematics Teaching in the Middle School* at **mtms@nctm.org**. Please include "Readers Write" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.

maintain responsibility for determining the correct idea. This is consistent with the notion from the *Professional Standards for Teaching Mathematics* (1995) that teachers must serve as the facilitator who must both filter while directing students' explorations as well as provide information while leading students. However, to move the classroom discourse to level 3, Mr. D. would need to work to establish an environment in which the students are filtering and pushing the conversation.

Episode Illustrating a Level 0/1

The interaction that occurred in the first episode showed an example of discourse when striving for level 2 in managing incorrect answers. In contrast, we would like to explore an excerpt in which managing an incorrect answer does not reach a level 2 or 3, to analyze what a teacher applying the Math Talk Framework might consider changing. In the following excerpt, Mr. D.'s class was practicing simplifying numerical expressions by working through a problem. Their task was to identify the mathematical properties they used in simplifying the expression on the left-hand side of the following identity:

$$4(6^2 \cdot \frac{1}{36}) = 4$$

First, one student suggested using substitution for $6^2 = 36$ to get

$$4(36 \cdot \frac{1}{36}) = 4.$$

After agreeing that this was correct, Mr. D. asked what comes next.

Excerpt 3

Anne: Wouldn't you distribute the four since it's outside?

Mr. D.: You wanted to use . . . you want to use the distributive property to distribute the four throughout?

Anne: Yeah.

Mr. D.: [Pause] Anne, come on down. [Mr. D.: writes

$$4\left(6^2 \cdot \frac{1}{36}\right) = 4$$

on the side board.]

Class: Aww! Oh!

Mr. D.: Let's work through this problem. Go ahead. I want you to rework distributing the four.

At this point, Anne came to the board to discuss her idea. As Mr. D. worked through the problem with Anne, he turned his back on the remainder of his class. While Anne and Mr. D. faced the board and engaged in a discussion that allowed Anne to see the error in her thinking, the students seated at their desks were staring off, flipping through notebooks, and talking. Once Anne believed she understood, she returned to her seat, and Mr. D. proceeded with the problem.

It is important to note here that Mr. D. never identified Anne's idea as being "wrong" or "incorrect." In fact, his decision to have her come to the board to discuss her idea had the potential to begin a classwide discussion about whether or not the distributive property was an appropriate step. However, the dialogue that occurred as a result of Anne's idea did not reach the whole class. Although he questioned Anne's thinking and used her errors for learning opportunities, his interaction with her to the exclusion of the rest of the class prevented them from asking questions of one another and building off the ideas of their peers. For these reasons, we believe this interaction reached level 1 of the Math Talk Framework.

It is a common mistake among middle school students to apply the distributive property whenever a constant is multiplied by a quantity within a set of parentheses. Thus, it was likely that other members of Mr. D.'s learning community agreed with Anne and believed applying the distributive property was an appropriate step. Although the discussion between Anne and Mr. D. helped to clarify Anne's thinking, it was unable to help other members of the community. As Anne and Mr. D. faced the board, their backs blocked the remainder of the community from seeing the work and hearing the discussion. As opposed to excerpt 2, in which silent members of the community could still be silent participants in the discussion, Mr. D. actually interfered with the participation of the seated students by excluding them from the conversation and limiting their opportunities to explain and justify.

EXPLORING ALTERNATIVE DECISIONS AND ACTIONS

We propose that the interaction observed in excerpt 3 could be changed to reach the entire community. We offer two alternatives that Mr. D. could have used in managing Anne's incorrect use of the distributive property. These alternatives explore how Anne's idea presented in excerpt 2 (**fig. 2**) could begin a level 2 interaction:

Anne: Wouldn't you distribute the four since it's outside?

Mr. D.: You wanted to use . . . you want to use the distributive property to distribute the four throughout?

Anne: Yeah.

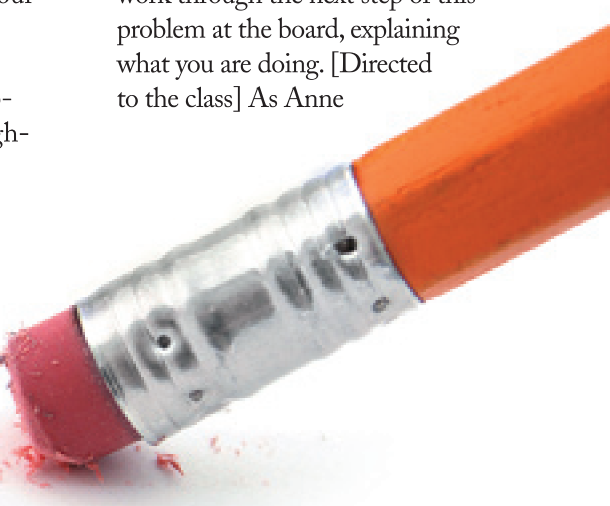
Alternative 1

At this point, instead of immediately bringing Anne to the board, Mr. D. could address the entire class, "Does anyone agree or disagree with Anne's idea?" Consistent with level 2, this question places responsibility for understanding on the students because it sets up the structure for building (Sherin, Louis, and Mendez 2000). By asking this question, Mr. D. communicates that he is not solely responsible for verifying the "correctness" of Anne's thinking. Instead, he directs the idea to the entire community for evaluation. However, from excerpt 1, we see that students do not always readily respond to this question, which in turn requires the teacher to persist in uncovering students' thinking. Mr. D. could prompt students to work in pairs on Anne's suggestion for a few minutes. With this additional time to think, students might contribute more to the discussion.

Alternative 2

On the other hand, Mr. D. could bring Anne to the board as he did in the original excerpt 3. To ensure that the entire community is able to participate in the evaluation and discussion of Anne's idea, Mr. D. could modify the discussion as follows.

Mr. D.: Anne, please use your idea to work through the next step of this problem at the board, explaining what you are doing. [Directed to the class] As Anne



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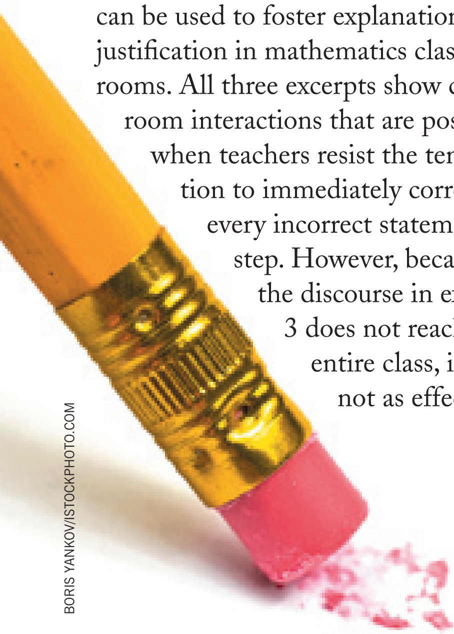
works, the rest of us should listen and come up with a question to ask her.

In this step, Anne is asked to justify her idea so that the rest of the class can understand her thinking. Mr. D. helps students to take an active role by encouraging them to listen to Anne and prepare a question to ask her. This encouragement prepares the way for student-to-student talk, which is an important component of level 2 and 3 interactions and a fertile environment for explanation and justification.

The alternatives suggested here are just two possibilities that could help manage Anne's incorrect use of the distributive property by holding the class responsible for understanding and evaluating the idea. The point here is that Mr. D. is not responsible for correcting Anne's thinking; instead, he shifts the authority so that it is shared with the entire class. As a result, students play more active roles in the discussion and learning and hold one another accountable for explanation and justification.

USING THE FRAMEWORK IN PRACTICE

The excerpts presented here displayed different ways that incorrect answers can be used to foster explanation and justification in mathematics classrooms. All three excerpts show classroom interactions that are possible when teachers resist the temptation to immediately correct every incorrect statement or step. However, because the discourse in excerpt 3 does not reach the entire class, it is not as effective as the discourse in excerpts 1 and 2. To effectively promote classroom discourse while managing an incorrect answer, a teacher should strive for a level 3 interaction that elicits the participation of every student through either verbal or silent participation.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The Math Talk Framework presents math teachers with a useful way to assess classroom discourse, whether or not it is in the context of managing incorrect answers. In particular, through the various levels, the framework allows teachers a means for determining whether students have enough responsibility for providing explanation and justification of their ideas and strategies. Using the excerpts and ideas presented in this article, teachers can determine ways to alter the discourse community so that it reaches level 3.

By fostering a classroom learning community at level 3, a teacher imparts a great deal of responsibility for learning on his or her students. Such responsibility includes constant explanation and justification of ideas, as well as an evaluation of the ideas of other members of the community. Thus, we believe classrooms that consistently operate at level 3 successfully foster explanation and justification, which lays an essential foundation for mathematical proof.

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