teaching children mathematics

Anatomy of a Good Review

What Is the Anatomy of a Good Review?

What makes a good review?

- The strengths and weaknesses of the article are indicated.
- The comments are as specific as possible.
- The mathematics and the pedagogy are examined carefully.
- The tone and substance of the review are respectful.
- The comments that you include are appropriate to forward to the author.

Just as you would guide your students, mention the strengths of the manuscript first and the weaknesses later. Frame the criticisms of the manuscript in a way that suggests steps that the author might take to improve the piece. *Each author needs specifics if he or she is to improve the content or style of the manuscript*. There is a place on the form to include confidential comments to the staff or Editorial Panel that you DO NOT want the author to see. However, what goes in the main review should be professional and offer constructive criticism, irrespective of the quality of the manuscript.

Although the Editorial Panel and journal editors will examine the mathematics, it is also the referee's responsibility to check the mathematics thoroughly. We make every effort to produce a high-quality, error-free journal; that goal can only be accomplished with careful attention to accuracy by everyone involved in the process.

What are your recommendation options?

You will be asked to select one of these three categories for your final recommendation: reject, revise, or accept.

REJECT: Recommend that a manuscript be rejected if you feel the topic or its treatment is already well known by the journal readers. If the author's ideas are difficult to understand or are not well executed and you feel that the author will be unable to improve the quality of the manuscript significantly, then it should be rejected. Also recommend REJECT if you feel that the ideas in the manuscript have the potential to make a contribution to the journal but the manuscript requires an extensive revision in style or organization.

REVISE: These manuscripts show strong promise but may need some stylistic improvement and some minor additions or deletions in the text. If you choose this option, the authors <u>**may be sent your comments**</u> and will be given a deadline by which to return the revision for a final review. The Editorial Panel may request that the revised manuscript be sent to you for a second review.

ACCEPT: Manuscripts in this category should address important topics in a clear, readable fashion. Minor editorial changes may be needed but these should be ones that the author can easily complete. Remember that comments from the referees are sent to the author as an aid in revising the article.

What does the review look like?

To help you get started, examine the items in this packet. They will give you guidelines for your review as well as an example of a published manuscript. The items include:

- Quick Reference Guide: Characteristics of a Helpful Review
- Manuscript Evaluation Form for Referees
- Case Study:
    - Part 1: Manuscript
    - Part 2: Referee Recommendations to Panel and Feedback to Author
    - Part 3: Editorial Panel Feedback to Author
    - Part 4: Published Article

Thank you for your interest in serving the mathematics education community by volunteering for this important task. If you have any questions, please e-mail the journal editor.

Teaching Children Mathematics: tcm@nctm.org.
Mathematics Teaching in the Middle School: mtms@nctm.org
Mathematics Teacher: mt@nctm.org

Quick Reference Guide: Characteristics of a Helpful Review

1. Read the review criteria and the questions on the evaluation form before reading the manuscript.

2. Summarize the intent and content of the article.

3. Provide a rationale for your recommendation to reject, revise, or accept.

4. Read the manuscript with these basic issues in mind:

   - Does the article add new information/ideas to the field or does it duplicate available materials?
   - Is the writing either too basic or too technical?
   - Did you identify any incorrect mathematics or inappropriate pedagogy?
   - Is the length adequate/appropriate for the topic?
   - Are all the figures necessary? Do the figures add something new to the text?
   - Is the manuscript appropriate for the audience? (Is it useful to teachers at various levels across grades, to mathematics supervisors, special educators, or teacher educators?)

5. Begin with positive comments. Point out both strengths and weaknesses of the article.

6. Include specific suggestions for improvements:

   - Correct mathematics as appropriate
   - Suggest deletions and/or additions
   - Consider reorganization of the manuscript as well as possible changes in title, subheadings, introduction, and conclusion.

7. Avoid personal biases.

8. Make comments that can be reproduced and shared with the author. Your comments and suggested revisions may be submitted in bulleted form. Where applicable, please refer to the manuscript page number and line number.

Teaching Children Mathematics
Manuscript Evaluation Form for Referees

1. For which audience is this manuscript appropriate?

a. Classroom Teacher:
- Preschool
- K-2
- 3-6
- K-6

b. University:
- Teacher Educator
- Preservice Teacher

c: Other:
- Teacher Leader
- Other

2. Select the appropriate ratings for this article:

a. Supports the mathematical development of children:

- Poor
- Fair
- Good
- Outstanding

b. Contributes to the professional or mathematical knowledge of teachers:
- Poor
- Fair
- Good
- Outstanding

c: Promotes a culture of equity in the teaching of mathematics:
- Poor
- Fair
- Good
- Outstanding

d: Quality of writing:

- o Poor
- o Fair
- o Good
- o Outstanding

2. Are there any mathematical errors or concerns?

- o Yes
- o No

If so, please explain

3. What is your recommendation for this article? (select one)

- o Reject
- o Revise and Resubmit
- o Accept

4. Would you be willing to work with the author(s) to get the manuscript into publishable form?

- o Yes
- o No

5. Regardless of your rating, please include the rationale for your evaluation below. Portions of your review may be sent to the author(s) in support of the Panel's decision.

Please respond to the following points in your review.

Summarize the article in a couple of sentences.

What are the strengths of the article?

What are the weaknesses?

What specific comments/suggestions might be shared with the author on how to improve the manuscript? Please focus comments on the substantive content and organization of the manuscript and not on mechanical errors.

Confidential comments for the Editorial Panel or NCTM staff (Not to be shared with the author(s).)

Other

CASE STUDY
Part 1:
Manuscript

TCM CASE STUDY
Part 1: Manuscript #09-08-159-3B

Encouraging Controversy in the Mathematics Classroom

Is the long side of an index card 5 inches or 7 inches? Do ten flats represent ten hundred or one thousand? Is 4 x 8 the same as 8 x 4, or are they different? To an adult, these may seem like trivial questions. How can an index card have two different lengths? Don't ten hundred and one thousand represent the same value? Aren't 4 x 8 and 8 x 4 both alike and different? As teachers, though, we recognize that these questions represent big mathematical ideas that young children do not necessarily find so straight forward. In fact, given the right setting these questions can lead to mathematical controversies within the classroom, with students arguing their different points of view. Controversies such as these can serve to ignite the mathematics classroom as they engage students in presenting and defending their stance on the issues at hand.

Mathematical controversies arise when a mathematical issue presents itself, sparking debate as students "take sides." Such controversies provide students with the opportunity to think about the mathematics in an effort to make sense of the situation and thereby make the correct choice. The proceeding discourse that surrounds the controversy allows the students to organize their own thoughts, formulate arguments, consider other students' positions, and communicate their own positions to their classmates. Such mathematical activity leads to internal "disequilibrium" for the learner, a term introduced by Piaget

(1970) and viewed as an essential component of learning. Recognizing that such controversies engage students in the processes of communication and reasoning and proof (NCTM 2000), teachers are often willing to be *reactive* in the sense that if a controversy occurs they will allow it to develop and encourage the discourse that surrounds it. But, is it possible to be *proactive* and actually facilitate the development of the controversy as a means for engaging students in these important mathematical processes? The purpose of this article is to share strategies for facilitating the development of controversy in the mathematics classroom. After a brief overview of the characteristics of issues that lead to controversy, three strategies will be described along with examples to support each strategy.

Characteristics of Issues that Lead to Controversy

As teachers who have purposefully planned to engage students in mathematical controversies, we have identified three characteristics of an issue that will allow it to become a controversy in the classroom. First, the issue is recognized by the students. If, for example, it comes in the form of two solutions that contradict (or seem to contradict) one another, the students view it as problematic that opposing solutions are being offered. Second, the issue is accessible to the students. Regardless of their mathematical backgrounds, all students are able to engage in the mathematics being discussed. Often, the availability of concrete manipulatives can provide this accessibility for some students who might otherwise not have access. Finally, the third characteristic is that the issue is debatable by the students. The students have the necessary background knowledge to debate the issue as they share the mathematical reasoning behind their views. Issues should not, for example, center around syntax or vocabulary as these are ideas that are not debatable.

Strategies for Encouraging Mathematical Controversies

Our work with a third grade class revealed three strategies that successfully facilitated the development of mathematical controversies. These strategies are described below along with a sample task or scenario to support the reader's understanding of the strategy.

Utilize Tasks that Reveal Students' Misconceptions

Tasks that are designed to reveal students' misconceptions provide the opportunity for controversies to arise. As an example, consider the broken ruler task, which is designed to reveal students' misconceptions regarding measurement of length. In this task, students were given a paper ruler like the one pictured in **figure 1** and asked to utilize it to measure the long side of an index card, which was 5 inches in length. Having read about using a broken ruler (Barrett, Jones, Thornton, & Dickson 2003), we anticipated that the task would lead to a controversy surrounding the index card's length, as some students would report the actual length (5 inches) but others would misread the ruler. As the lesson unfolded, three separate controversies arose. First, some students reported that the index card measured 5 ½ inches AND 6 inches, leading to the controversy of whether or not the long side of the index card could have 2 different lengths. Second, as students provided their arguments regarding the possibility of 2 different lengths, the controversy of whether to count the lines on the ruler or the spaces in between the lines arose. Finally, the controversy of where to line up the index card, either at the initial line or the end of the ruler, surfaced.

Figure 2 contains dialogue from the initial controversy dealing with whether or not an object can have two different lengths. From the students' responses, one can see that by utilizing a task that would reveal students' misconceptions, a controversy arose enabling students to share their ideas and defend their reasoning. In doing so, they clearly have met the expectations of the Process Standards (NCTM 2000).

Design Writing Prompts that Force Students to Choose a Side

A second strategy for facilitating the development of a controversy in the math classroom is designing writing prompts that force students to choose a side. Having begun our multiplication unit by looking at groups of different sizes, we very quickly found ourselves considering the role of the factors in an expression such as 4 x 8. Students were given the writing prompt in **figure 3**.

Figure 4 displays three students' initial responses to the prompt. In **figures 4a** and **4b**, students indicated that these expressions are the same because they have the same product. In **figure 4c**, the student wrote that the two expressions were different, basing his decision on the representations of each. By providing the students with the opportunity to write first, they had time to think through their ideas before listening to their classmates. Through the expression of different opinions, the controversy arose, but more importantly students engaged in communicating their reasoning to their classmates. In the initial discussion, the prevalent viewpoint was that the two expressions were the same, since they had the same answer. Note that in **figure 4c** the student has erased the word "not" in his sentence after hearing a fellow student's argument. As the class discussion progressed, students began representing the two expressions with equal-sized groups[1] as a means of demonstrating that the two expressions were different.

The day after the controversy, students were given the opportunity to respond to the prompt again. **Figure 5** contains the follow-up writings of the students from figure 4. Notice that in each case, the student has recognized the difference between the representations of 4 x 8 and 8 x 4. Notice that the student if **figure 5b** writes that the two expressions are both different and alike.

Ask Open-ended Questions

A third strategy for facilitating mathematical controversies is asking open-ended questions. From a planning perspective, it is sometimes difficult to know ahead of time whether the discussions that arise from such questions will lead to a controversy in the classroom. If, however, teachers utilize their previous experiences in working with students they can anticipate with great accuracy how students will respond and therefore anticipate the controversy that will arise.

Take, for example, the question posed in **figure 6**. In previous lessons, the third-grade students in our classroom had represented and decomposed 3-digit numbers with base-10 blocks. Therefore, this seemed like a natural question for them to consider. The controversy occurred when the students had to decide whether the 10 flats were equivalent to ten hundred or one thousand, an issue that arose through their own sharing of ideas. As an adult, the issue of ten hundred versus one thousand may not seem like an issue at all. In fact, one might feel it is just an issue of syntax or vocabulary, resulting in treating this as a non-issue. With this view, the adult's tendency would be to just say the 10 flats represent one thousand which has a value of ten hundred. Doing so would not help students like Kendra, though, who sees one thousand as being much larger than ten hundred. The issue is not in how to read the number 1000. Instead, the mathematics underlying this issue includes the understanding that a value can be represented and expressed in multiple ways as well as the structure of the base-10 system that allows for 10 of one unit (in this case 10 hundreds) to be equivalent to 1 of another unit (in this case 1 thousand). These ideas around place value are key mathematical concepts and therefore worthy of attention.

Conclusion

As teachers, it is most likely the case that we utilize tasks, writing prompts, and questions similar to those presented here. It is not the case, however, that we always recognize the potential that these have for evoking mathematical controversies. As demonstrated in the previous examples, by encouraging controversy we are engaging students in important mathematical processes; namely reasoning, proof, and communication. These processes in turn promote learning. Although learning is clearly important, just as important is the classroom atmosphere that is created that allows learning to occur. When students are presenting and defending their ideas, math class becomes fun for everyone!

Footnote

[1] The students in this classroom often utilized circles and stars for representing multiplication with equal-sized groups. For example, 4 x 8 would be represented with 4 circles, each containing 8 stars. See Burns (2001) for an explanation of the game “Circles and Stars.”

References

Barrett, J. E., Jones, G., Thornton, C., & Dickson, S. Understanding Children's Developing Strategies and Concepts for Length. In Clements, D. H. (Ed), *Learning and Teaching Measurement*. Reston, VA: NCTM, 2003.

Burns, M. *Teaching Arithmetic: Lessons for Introducing Multiplication, Grade 3*. Sausalito, CA: Math Solutions Publications, 2001.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Piaget, J. *Science of Education and the Pyschology of the Child*, translated by Derik Coltman. New York: Orion Press, 1970.

Figure Captions

Figure 1. Broken ruler used for measuring.

Figure 2. Vignette of the broken ruler controversy.

Figure 3. Writing prompt for multiplication.

Figure 4. Student writing samples prior to controversy.

Figure 5. Student writing samples following the controversy.

Figure 6. Student arguments from the ten flats controversy.

Figure 1

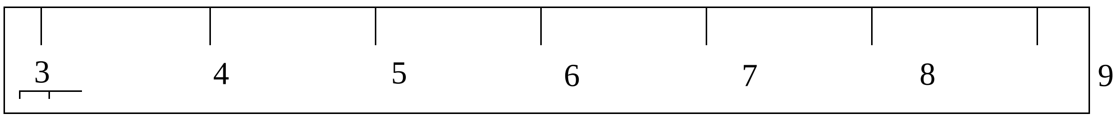


Figure 2

Frank: There's two answers.
Teacher: Is it possible for something to have two different lengths? Outside of a piece of gum that can be stretched . . . If you can't stretch it, is it possible for something to have two different lengths?
Jenaria: Like what Brianna said like you can either round and it can go up to 6 inches or you don't have to round. You can just leave it like it is at 5 ½ inches or you can round it and have it be 6 inches.
Karl: Yes, if you do it like that (motions to 2 sides of the index card). . . . (After clarification that the students were just measuring the long side of the card.) Oh, no. Because the ruler only can be like (pause) if you do it like that you've still got the same thing. (pointing toward the numbers on the ruler). Only if you turn it around. **Teacher:** So that changes the length of the card when you move the ruler.
Karl: Not exactly.
Frank: (Using the broken ruler to demonstrate at the front of the class, Frank aligns the end of the index card with the end of the ruler.) Now, there's 1, 2, 3, 4, 5, and a half. (Afterwards, he slides the index card so that it aligns with the 3-inch mark.) But now, um, there is 1, 2, 3, 4, 5, 6. (Note, that Frank is counting the inch marks on the ruler and not the spaces in between them.) You didn't need to move the ruler because this is in centimeters.
Karl: [I disagree with] him doing the centimeters. We are not learning about centimeters. . . .That's mostly the only way you can do it two times. That's the only way you can get 2 different measurements – using centimeters and inches.
Ayona: If he changes the ruler to here (aligning the edge of the index card with the edge of the broken ruler) then you have to count this part too (motioning to the initial part of the index card before the 3-inch mark). **Frank:** Yes, I was counting this (pointing toward the part of the card after the final inch mark); it's a half.
Ayona: But you weren't counting this (pointing toward the initial part of the index card before the 3-inch mark). **Frank:** Oh yeah.

Figure 3

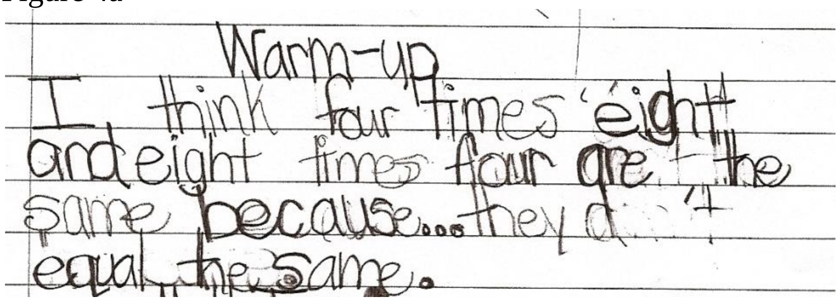
4 groups of 8	8 groups of 4
4 x 8	8 x 4

In your journal, complete this statement.

I think 4 x 8 and 8 x 4 are / are not the same because . . .

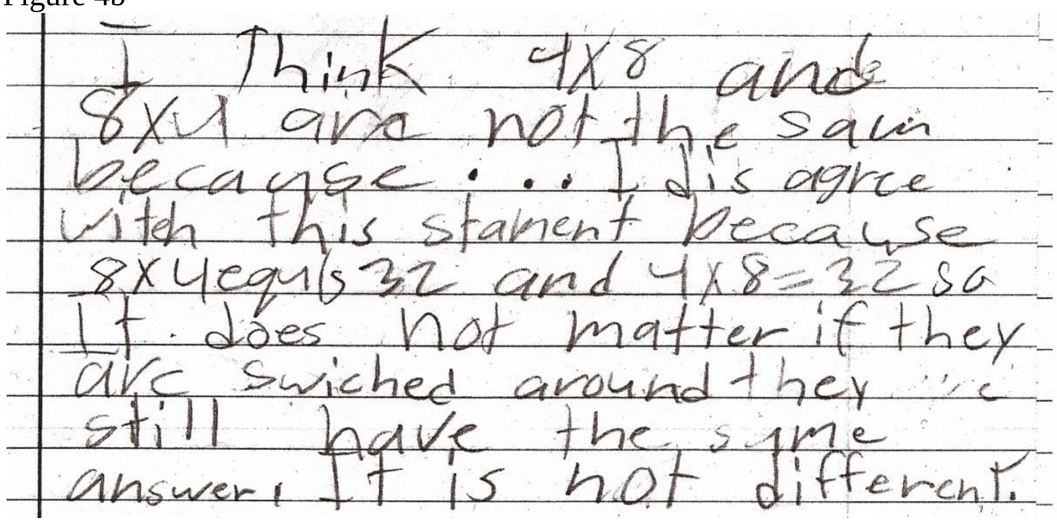
Figure 4

Figure 4a



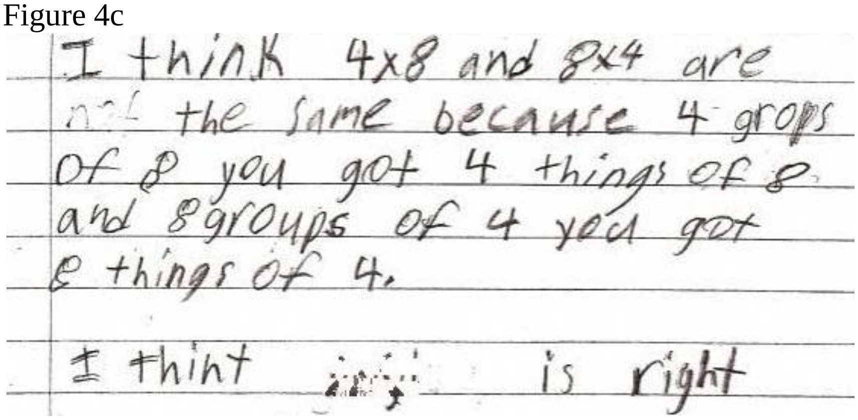
Warm-up
I think four times eight and eight times four are the same because... they equal the same.

Figure 4b



I Think 4X8 and 8X4 are not the sam because . . . I dis agree with this stament because 8X4 equls 32 and 4X8=32 so It does not matter if they are swiched around they still have the same answer. It is not different.

Figure 4c



I think 4x8 and 8x4 are not the same because 4 grops of 8 you got 4 things of 8 and 8groups of 4 you got 8 things of 4.

I thint [illegible] is right

Figure 5

Figure 5a

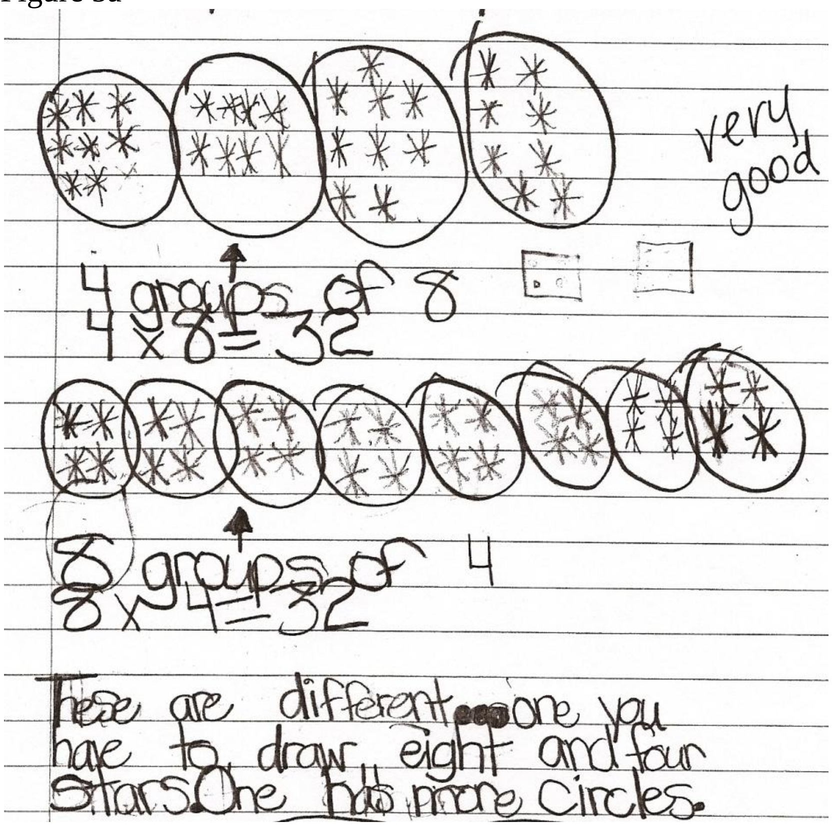


Figure 5b

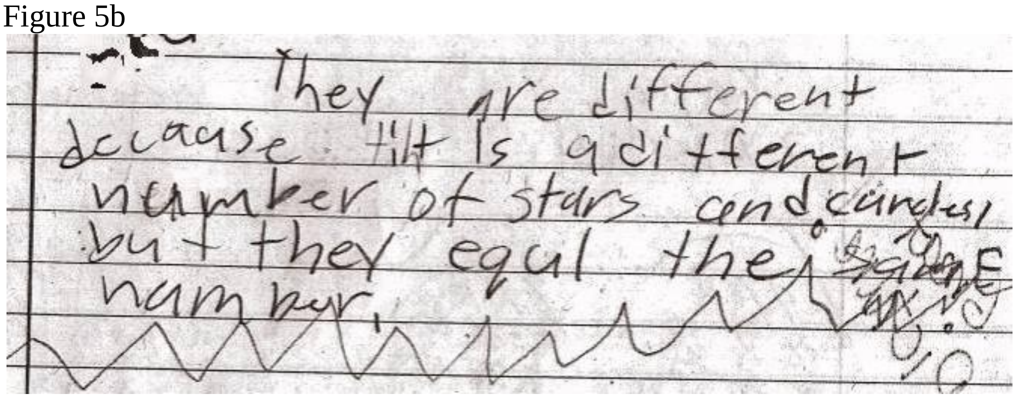

They are different decause it is a different number of stars and circles, but they equl the same number,

Figure 5c

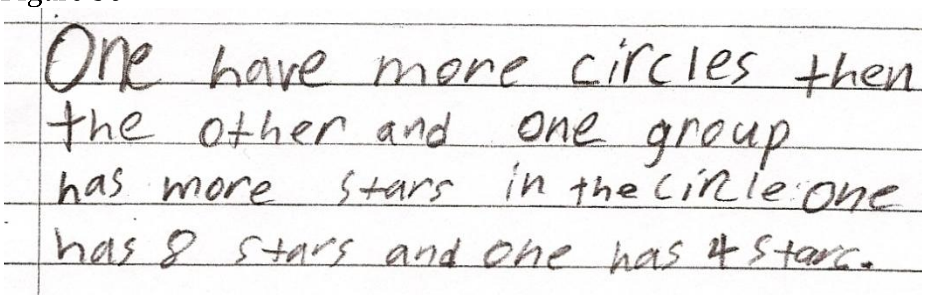

One have more circles then the other and one group has more stars in the cirle one has 8 stars and one has 4 stars.

Figure 6

Teacher: If 10 units can be put together to form 1 long, and 10 longs can be put together to form 1 flat, what do we have when we put 10 flats together? *After requiring the students to think inside their heads and then share with a partner . . .*

Alan: You think it's ten hundred but it can't go above ten hundred so I'd have to all go to one thousand. . . . So, I mean, it would start over but at a higher level, one thousand.

Doug: (Standing at the board) The way how I did it, it had these zeroes and one one (Writes "1000" on board). And, and, and after- (trying to get the attention of the students) Look. And after there are three numbers, I put a comma (puts a comma between the 1 and first zero). That's what I thought. . . . Because, if it – if he thought it was ten hundred (erases the comma), that wouldn't make sense. . . . Nope, that would be wrong (X's out the "1000" on the board and writes "1,000" out to the right). But, this is right (puts a check mark by the "1,000").

Kendra: I want to say I disagree with all of them because if you count on your fingers you're gonna get one hundred, two hundred, three hundred, four hundred, five hundred, six hundred, seven hundred, eight hundred, nine hundred, ten hundred. How in the world can you skip from all the way to ten hundred…From one hundred to a thousand? . . .When you count on your fingers, when you count on your fingers, you're going to get to ten hundred and that's how I know because I know I can't skip all the way from one hundred to a thousand. I ain't got that many fingers.

James: Um, I thought that it's, it's actually both of them because, because ten equals (pause) . . . (Looking at "1000" and "1,000" on the board) . . . all of the things that had changed for both of them was that comma because they've got the – they've got the same number of zeroes and one. The only thing that would really actually change was if he putted that comma right there after the one then it would have been the same.

CASE STUDY
Part 2:
Referee
Feedback

(Referee/Reviewer #3)

Overall Recommendation	Accept
Classroom Teacher:	K-6
University	Preservice Teacher
Other	Teacher Leader
Importance of topic	**Outstanding/Good**
Quality of ideas	**Outstanding**
Quality of writing	**Outstanding**
Would you be willing to work with the author(s) to get the manuscript into publishable form?	Yes
Are there any mathematical errors or concerns?	No

Note	Comment
Mathematical errors comments	
Summarize the article in a couple of sentences	This is a cogent, well-written article about using controversy as a vehicle for mathematical communication and reasoning.
What are the strengths of the article?	The writing is excellent. It is easy to follow and has sample student responses to illustrate their technique. The controversy technique is the mathematical equivalent of presenting discrepant events to students in science education. It is an easy way to promote discourse with your students. It's a great idea because it not only encourages discourse, mathematical communication, and reasoning, but is a good way to determine what individual students understand about the topic at hand.
What are the weaknesses?	I didn't find any.
Summarize your specific suggestions for the author	Line 59- Replace "one" with "you" to reach the readers. Line 89- Insert comma between "students" and "they." Line 113- Replace 2nd "that" with "which" to eliminate using "that" twice.
Confidential comments for the Editorial Panel or NCTM staff	This is the first article I've reviewed that was ready to be published with a just couple of grammar changes.
Other	

(Referee/Reviewer #4)

Overall Recommendation	Revise and Resubmit
Classroom Teacher:	3-6
University	
Other	
Importance of topic	**Good/Fair**
Quality of ideas	**Fair**
Quality of writing	**Good**
Would you be willing to work with the author(s) to get the manuscript into publishable form?	No
Are there any mathematical errors or concerns?	Yes

Note	Comment
Mathematical errors comments	4x8 = 8x4 They are not different and should not be considered different. Commutative property is the emphasis.
Summarize the article in a couple of sentences	It is an interesting approach to viewing commutative property and/or understanding different approaches to weighing in similarities and differences in problems but it should not be considered as a controversial issue rather a connection to the properties.
What are the strengths of the article?	It is a good idea to generate an interest to various approaches of doing math and seeing different approaches to the end product as long as all students are as equally involved as possible to meet their learning requirements.
What are the weaknesses?	It is in actual fact difficult to implement or generate discussion with all younger audiences without considering the diverse learners. They may be left behind if participation is problematic to them and also consideration needs to be given to different learning styles.
Summarize your specific suggestions for the author	The author needs to consider all learning styles and state how or where inclusion occurs. Consideration for the ethnically minority students is necessary.
Confidential comments for the Editorial Panel or NCTM staff	This is an interesting approach but above mentioned factors need to be considered as part of revision. According to the current submission it is unlikely all children are considered in the discussion.
Other	

CASE STUDY
Part 3:
Editorial Panel Feedback

The Panel member's comments that were forwarded to the author:

The authors provide a practical approach to creating opportunities for mathematical argumentation in an elementary classroom. The three examples used demonstrate the difficulties and misconceptions children have about seemingly straightforward topics. Giving children the opportunity to discuss the differences in perspective reveals valuable information about children's understanding. With revision the article may have the potential to make an important contribution to TCM. Please consider the following questions and suggestions for revision:

* Line 2: I had to read the first sentence several times and I'm still not sure what was meant. Did you mean 5 inches 'or' 7 inches?

* I'd like to see the introduction reworked. I liked how the manuscript starts with three questions, but since I didn't understand the first question and the third question is not easily answered when given in context (rather than abstractly), I didn't see them as trivial questions. Consider focusing the introduction on children's attempts to understand mathematical concepts and how controversies/different opinions surface in the course of a class. Stress that rather than halting those discussions (to provide the 'real' answer) use it as an opportunity to understand children's difficulties and expand their thinking.

* Where does the word "controversies" come from? I don't think the controversies are necessarily instances of "disequilibrium". Disequilibrium (according to my interpretation of Piaget) is when a child's experience through interaction with the world or others leads to something unexpected and conflicts with previous understanding. Having an argument with someone doesn't create disequilibrium (if the person doesn't experience any conflict or confusion in the two sides of the argument). The authors' description of the phenomena sounds more like mathematical argumentation. Please consider the work of M. Lampert, E. Yackel, G. Krummheuer. For example, you may want to start with:

Yackel & Hanna (2003). Reasoning and proof. In J. Kilpatrick et al. (Eds). A research companion to Principles and Standards for School Mathematics. Reston, VA: NCTM

By drawing on the argumentation literature you may be able to make stronger connections to how these controversies are important for learning.

* Section starting on line 26: Since an example has not been presented at this point, it is difficult to make sense of the three characteristics. Perhaps you can point to the features after an example. For example, take the 4x8 issue. The authors indicate that it is recognizable, accessible and debatable ... correct? Can you connect these features to this particular issue? Also, how are recognizable and debatable different? Was the issue recognized by the students or given to the students as a debatable issue.

* The various strategies for encouraging controversies are valuable. I would like to see more examples within each of the categories. The additional examples provided do not need student responses, but even a list of questions might be useful. For example, the authors indicate that teachers could design prompts that force students to choose a side. Could they give a list of other possible prompts?

Asking open-ended questions sounds a bit vague. I can think of open-ended questions that don't prompt controversy. What qualities of an open-ended question are necessary? Could they provide other examples?

* All three classroom examples provided are quite interesting. I would like to see more of a connection to why having students engage in controversy is important within those contexts. All examples seem to end after both sides of the argument are clear, but I don't see how that helped students overcome the issue and learn from it. For example, what happened after the students' journal writing of the 4x8 issue? It seems as though there was an opportunity to discuss it in the class, but the authors don't appear to write about it. Also, what is the teachers' role during these debates? What if neither side is able to see the other side of the argument?

Is the third example an open-ended question? The question, "What do we have when we put 10 flats together?" is not open-ended. The children could have simply answered 1000 and moved on.

* I'd like to see a stronger conclusion. My experience with arguments in class is not that it is "fun for everyone" and can, in fact, be quite painful if students feel their ideas are being attacked. The authors may either have to provide more information on the emotional aspect of argumentation or reword the conclusion to emphasize the importance of developing a classroom environment that values risk-taking, open discussion, debating ideas, etc.

I am hopeful that the authors will consider the questions and suggestions provided above as they revise the manuscript.

[NOTE: Subsequent manuscript revisions and comments are not included for the purpose of this case study.]

CASE STUDY
Part 4:
Published Article

3 strategies for promoting math disagreements

Engage your students in reasoning and sense making with these effective instructional plans.

By Angela T. Barlow and Michael R. McCrory

As elementary school students attempt to understand mathematical concepts, engaging in the processes of reasoning and sense making is important (Martin and Kasmer 2009–2010). To do so, students should be expected to listen to and challenge their classmates' ideas (Yackel 2001). Disagreements provide students with the impetus to think deeply about mathematics in an effort to make sense of a situation. The discourse that surrounds the disagreement allows students to organize their thoughts, formulate arguments, consider other students' positions, and communicate their positions to their classmates. As differing opinions surface during classroom discussions, teachers receive valuable insights that help them understand children's difficulties with what might otherwise seem like straightforward math. Additionally, as students begin debating mathematical ideas, teachers have occasion to expand students' mathematical thinking.

Recognizing that mathematical disagreements among students engage them in the processes of reasoning and sense making (Yackel 2001), teachers are often willing to be reactive in the sense that if a disagreement occurs, they will allow it to develop and will encourage the discourse that surrounds it. But

VANDA GRIGOROVIC

FIGURE 1

After a weekend to think about the role of factors in multiplication, students were given this writing prompt.

4 groups of 8	8 groups of 4
4×8	8×4

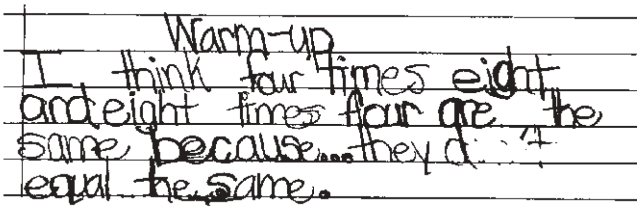
In your journal, complete this statement:

I think 4×8 and 8×4 are/are not the same because . . .

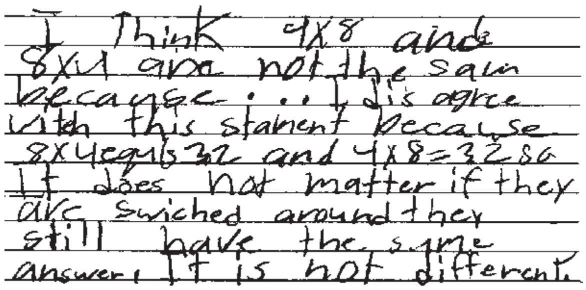
FIGURE 2

Students did writing samples before their multiplication disagreement, expressing their opinions and justifying them.

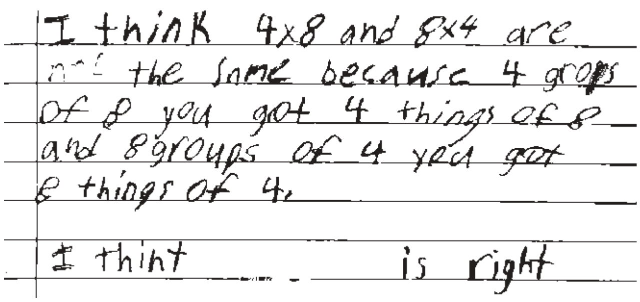
(a) The student self-corrected a journal entry before the classroom discussion.



(b) This student reflected understanding with a sound argument that supported the mathematical reasoning.



(c) One student revised his statement after hearing a classmate's argument.



is it possible to be proactive and actually facilitate the development of a disagreement as a means for engaging students in these important mathematical processes?

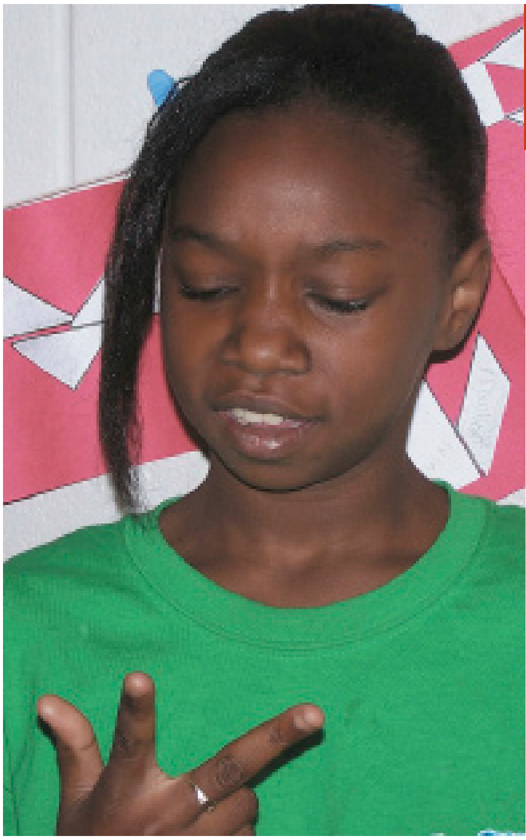
Strategies for encouraging mathematical debate

Our work with a third-grade class revealed three instructional strategies that create opportunities for students to engage in mathematical disagreements in the classroom. The paragraphs that follow describe these strategies, as well as sample tasks or scenarios and student artifacts to support the reader's understanding of each strategy. Additionally, we describe characteristics of issues that lead to disagreements.

1. Force students to choose a side

Writing prompts that force students to pick a side facilitate their engagement in a mathematical disagreement. Choosing sides naturally elicits opportunities for disagreement. We used this strategy to facilitate a discussion during a unit that focused on modeling multiplication. Having begun our multiplication unit by looking at groups of different sizes, students quickly found themselves considering the role of the factors in a multiplication expression. In fact, at the end of class on a Friday, Doug said that 4×8 and 8×4 were the same. Jenaria quickly disagreed. We directed students to think about this disagreement over the weekend. At the beginning of the next class period, they received a writing prompt (see **fig. 1**).

Of three students' initial responses to the prompt, two students indicated that these expressions are the same because they have the same product (see **figs. 2a** and **2b**). The third student wrote that the two expressions are different, basing his decision on the representations of each (see **fig. 2c**). Having the opportunity to write first gave students time to think through their ideas before listening to their classmates. As they displayed their writings via a document presenter, a disagreement arose, engaging students in communicating their reasoning to their classmates. In the initial discussion, the prevalent viewpoint was that the two expressions are the same since they have the same product. In **figure 2c**, the student erased the words *not* and *don't* in his sentence after hearing a classmate's argument. As the class discussion progressed, students began representing the two expressions with equal-sized groups as a means of demonstrating that the two expressions are different. Students in this classroom had often used circles and stars to represent multiplication with equal-



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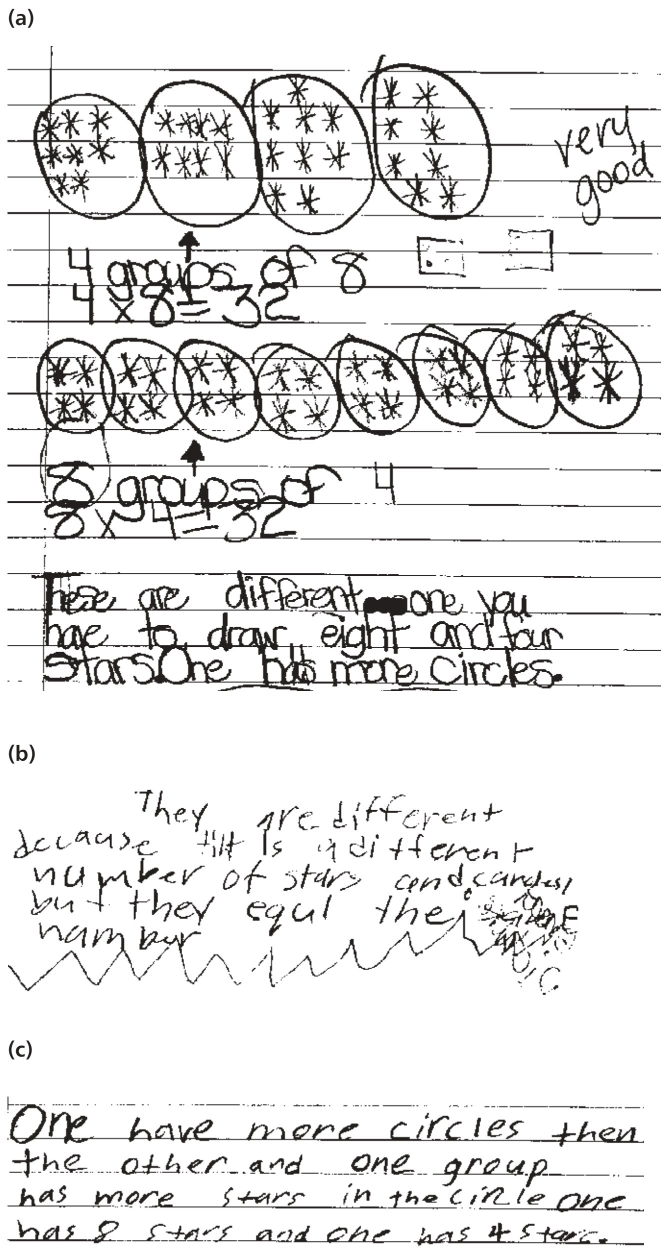
Giving students opportunities to choose a side and write their ideas sets the scene for math disagreements, dialogue, reasoning, and sense making.

FIGURE 3

Writing samples following the multiplication disagreement show that students now realized that the order of factors does not affect the answer but does affect the representation.



size groups. For example, 4 × 8 would be represented with four circles, each containing eight stars. (See Burns [2001] for an explanation of a game using circles and stars.)

The day after the controversy, students had a chance to respond to the prompt again (see **fig. 3**). In each case, the student recognized the difference between the representations of 4 × 8 and 8 × 4. One student wrote that the two expressions are both different and alike (see **fig. 3b**). After everyone had completed their writing, randomly selected students once again displayed their writings via the document presenter. As a result of this disagreement, students realized that the order of the factors does not impact the answer; but it does impact the representation.

In the previous multiplication example, the relationship of the two expressions provided two seemingly opposing sides to the issue (i.e., either the two expressions are the same, or they are different), furnishing students with the chance to state and support their reasoning. The writing prompt gave all students the chance to think about the issue and select a stance to take before the disagreement ensued. Two similar writing prompts follow:

- **Mrs. Jones** gave her students the equation 7 + 8 = ___ + 5 and asked them to tell what number should go in the blank to make the equation true. Mary said that a 15 should go in the blank. Keyon said that a 10 should go in the blank. Who do you agree with, and why?
- **Jamie** said that a square is a rectangle. Do you agree with Jamie? Why, or why not?

Once students have responded to the writing prompt, they should have opportunities to share

FIGURE 4

Such tasks as using a broken ruler for measuring are designed to uncover misconceptions and elicit disagreements.

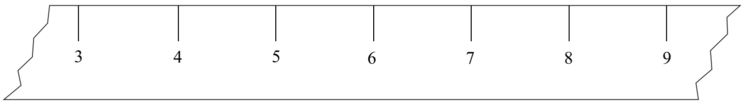


FIGURE 5

Had disagreements between students not occurred, gaps within their understanding most likely would not have been discovered. As students demonstrate their viewpoints to their peers, they make connections between their ideas, and they resolve other false ideas. (Teacher prompts are in bold text.)

[*Frank*] There's two answers.

Is it possible for something to have two different lengths? Outside of a piece of gum that can be stretched . . . If you can't stretch it, is it possible for something to have two different lengths?
[*Jenaria*] Like what Brianna said; like you can either round, and it can go up to six inches, or you don't have to round. You can just leave it like it is at five-and-a-half inches, or you can round it and have it be six inches.

[*Karl*] Yes, if you do it like that. [*He motions to two sides of the index card, pauses, then clarifies that the students were just measuring the long side of the card.*] Oh, no. Because the ruler only can be like [*pauses*] if you do it like that, you've still got the same thing [*pointing toward the numbers on the ruler*]. Only if you turn it around.

So that changes the length of the card when you move the ruler.
[*Karl*] Not exactly.

[*Using the broken ruler to demonstrate at the front of the class, Frank aligns the end of the index card with the end of the ruler.*] Now, there's 1, 2, 3, 4, 5, and a half. [*Afterward, he slides the index card so that it aligns with the three-inch mark.*] But now, um, there is 1, 2, 3, 4, 5, 6. [*Note that Frank is counting the inch marks on the ruler and not the spaces between them.*] You didn't need to move the ruler, because this is in centimeters.

[*Karl*] [I disagree with] him doing the centimeters. We are not learning about centimeters. That's mostly the only way you can do it two times. That's the only way you can get two different measurements—using centimeters and inches.

[*Ayona*] If he changes the ruler to here [*aligning the edge of the index card with the edge of the broken ruler*], then you have to count this part too [*motioning to the initial part of the index card before the three-inch mark*].

[*Frank*] Yes, I was counting this [*pointing toward the part of the card after the final inch mark*]. It's a half.

[*Ayona*] But you weren't counting this [*pointing toward the initial part of the index card before the three-inch mark*].

[*Frank*] Oh, yeah.

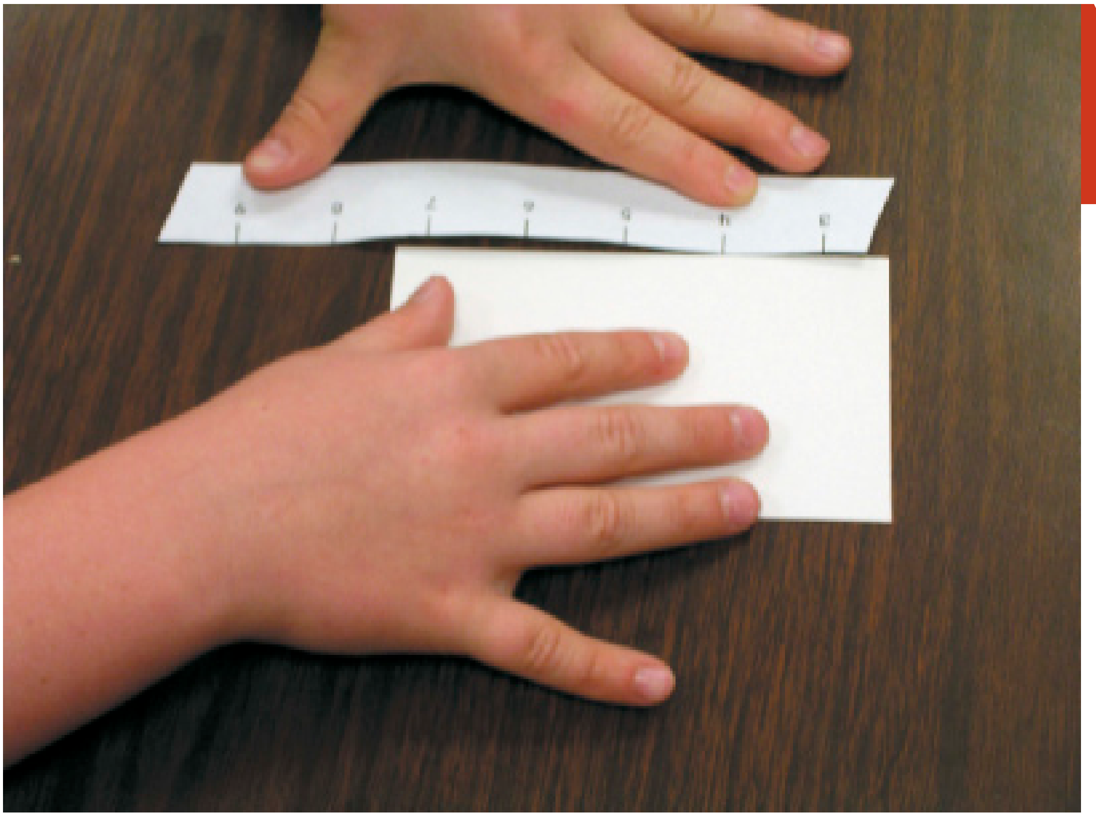
and support their reasoning, thus engaging in the disagreement.

2. Reveal students' misconceptions

Tasks that are designed to address students' misconceptions provide the opportunity for disagreements to arise. As an example, consider the broken-ruler task, which is intended to reveal students' misconceptions regarding measurement of length. In this task, students were given a paper ruler (see **fig. 4**) and asked to use it to measure the long side of an index card. Having read about using a broken ruler (Barrett et al. 2003), we anticipated that the task would lead to a disagreement surrounding the index card's length, as some students would report its actual length (five inches), but others would misread the ruler. As the lesson unfolded, three separate disagreements arose. First, some students reported that the index card measured five-and-a-half inches *and* six inches, leading to a disagreement over whether the long side of the index card could have two different lengths (see **fig. 5**). Second, as students provided their arguments regarding the possibility of two different lengths, a disagreement arose over whether to count the lines on the ruler or the spaces between the lines. Finally, a disagreement surfaced over where to line up the index card, either at the initial line or at the end of the ruler.

Examining students' responses to this example reveals that using a task designed to uncover their misconceptions prompts disagreements that give students the chance to share their ideas and defend their reasoning. Student discussions generated by the disagreement reveal an interesting possibility: When measuring lengths, students can actually have different results if dissimilar units are used in the measuring process. According to Karl, "That's the only way you can get two different measurements—using centimeters and inches." Such a revelation facilitated thinking about the underlying assumption of the disagreement, that we were all using the same units.

In a task adapted from Schifter (1999) and Clements and Sarama (2000), students identify which figures are triangles and give a description of how they know whether the figure is a triangle (see **fig. 6**). On the surface, any disagreement that comes from this task may appear to be centered on math vocabulary or the definition of a triangle.



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Discussing the broken-ruler task revealed the unspoken assumption that everyone was using the same units of measurement (inches).

Elementary school students, however, have not typically reached the descriptive level, where they can characterize shapes by their properties. So they rely on precognition, or visual-level thinking, where shapes that "look" like triangles can be classified as triangles (Clements and Sarama 2000). Thus, without the students having made this transition from precognition, or visual-level thinking, to a descriptive level of thinking, this task helps students develop their own understandings of what a triangle is and is not by allowing them to disagree on particular figures, formulate mathematical arguments for their points of view, and carry out a debate until the figure can be properly classified and agreed on.

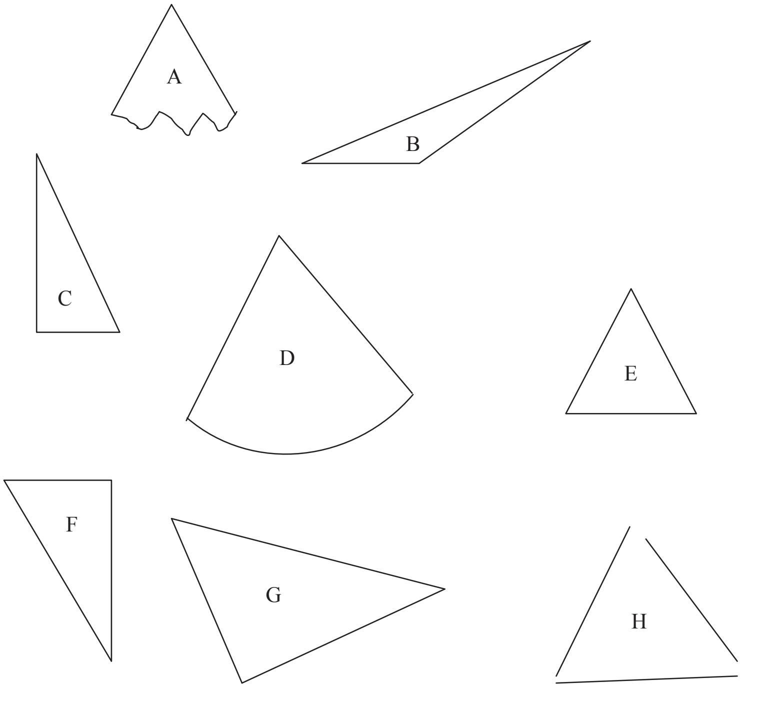
3. Recall last year's disagreements

A third strategy for facilitating mathematical disagreements is to purposefully design and ask questions that have led to disagreements in your previous classes. For example, because the third graders in our classroom had experience representing and decomposing three-digit numbers with base-ten blocks, we posed what seemed like a natural question for students to consider (see **fig. 7**). The disagreement occurred when students had to decide whether the ten flats were equivalent to ten hundred or one thousand, an issue that arose through sharing their ideas. For adults, ten hundred versus one thousand may not seem like an issue at all. In fact, one might

FIGURE 6

The triangle-classification task adapted from Schifter (1999) and Clements and Sarama (2000) prompts students to debate until they can properly classify a figure and agree on it.

Directions: Examine each two-dimensional figure below. Draw a circle around the figures that are triangles. At the bottom of the page, write a description of how you can tell whether a figure is a triangle.



feel it is just a question of syntax or vocabulary. With this view, the adult's tendency would be to say that ten flats represent one thousand, which has a value of ten hundred. Doing so would not help students like Kendra, though, who see one thousand as being much larger than ten hundred. The issue is not in how to read the number 1000. Instead, the mathematics underlying this issue includes understanding that a value can be represented and expressed in multiple ways as well as understanding that the structure of the base-ten system allows for ten of one unit (in this case, ten hundreds) to be equivalent to one of another unit (in this case, one thousand). These ideas around place value are key mathematical concepts and therefore worthy of attention.

In the example in **figure 7**, James's reasoning convinced the class that one thousand and ten hundred are equivalent. Follow-up tasks and discussions over the next few days engaged students in representing four-digit numbers in multiple ways, further building on these ideas.

Prior to asking the question, we really did not anticipate a disagreement. Looking back, however, we now understand why one arose. We are thankful that the question facilitated its occurrence. Now that we are aware of this issue, we can plan to use this same question in future classes, recognizing the importance of the disagreement that it will most likely invoke.

FIGURE 7

These third graders had previously represented and decomposed three-digit numbers with base-ten blocks. Arguments arose around place value—a key mathematical concept that is worthy of attention. (Teacher prompt is bold.)

If ten units can be put together to form one long, and ten longs can be put together to form one flat, what do we have when we put ten flats together? [*The teacher instructed students to think inside their heads and then share with a partner*.]

[*Alan*] You think it's ten hundred; but it can't go above ten hundred, so I'd have to all go to one thousand. So, I mean, it would start over but at a higher level, one thousand.

[*Doug, standing at the board*] The way how I did it, it had these zeroes and one 1 [*writing* 1000 *on the board*]. And, and, and after [*trying to get the other students' attention*], look. And after there are three numbers, I put a comma [*placing a comma between the* 1 *and the first zero*]. That's what I thought. Because, if it—if he thought it was ten hundred [*erasing the comma*], that wouldn't make sense. Nope, that would be wrong [*Xing out the* 1000 *on the board and writing* 1,000 *out to the right*]. But, this is right [*putting a check mark by the* 1,000].

[*Kendra*] I want to say I disagree with all of them, because if you count on your fingers, you're gonna get one hundred, two hundred, three hundred, four hundred, five hundred, six hundred, seven hundred, eight hundred, nine hundred, ten hundred. How in the world can you skip from all the way to ten hundred? From one hundred to a thousand? When you count on your fingers, when you count on your fingers, you're going to get to ten hundred, and that's how I know, because I know I can't skip all the way from one hundred to a thousand. I ain't got that many fingers.

[*James*] Um, I thought that it's, it's actually both of them because, because ten equals [*pausing to look at* 1000 *and* 1,000 *on the board*] all of the things that had changed for both of them was that comma because they've got the, they've got the same number of zeroes and one. The only thing that would really actually change was if he putted that comma right there after the one. Then it would have been the same.

Issues that lead to disagreements

As teachers who have purposefully planned to engage students in mathematical debate, we have identified three significant characteristics of issues that facilitate students' engagement in disagreements in elementary school classrooms.

1. Center on a mathematical concept

The issue must center on a mathematical concept. Issues such as syntax, math vocabulary, and procedures tend to be supported by the basic rules of mathematics and should not be up for debate. On the other hand, a group of students often do not understand math concepts in the same way; these differences can lead to a situation where a disagreement arises. This type of situation gives the teacher a prime opportunity to allow students to clear up their own misunderstandings by internalizing the results of the discussion and assimilating this new, self-remediated understanding into their mathematical cognition. For example, the disagreement during the triangle-identification task causes students to evaluate their understanding of what constitutes a triangle and develop those characteristics into a working definition of a triangle. These changes in thinking will result in the migration of students' thinking processes from precognition and visual levels to the descriptive level.

2. Are accessible to all

Second, the issue must be accessible to all students in the classroom. Regardless of their mathematical misunderstandings around the disagreement, students should have some true understanding that will serve as a foundation for their position on the disagreement. This foundation allows students to engage in the math being debated and provides the teacher with a common point of knowledge among the students as he or she facilitates the discussion. In the multiplication example, students' access to a means for representing multiplication allowed them a point of entry for investigating the role of the factors in a multiplication expression. Although concrete materials were not used in this case, often the availability of manipulatives can provide this accessibility for some students who might otherwise not have access.

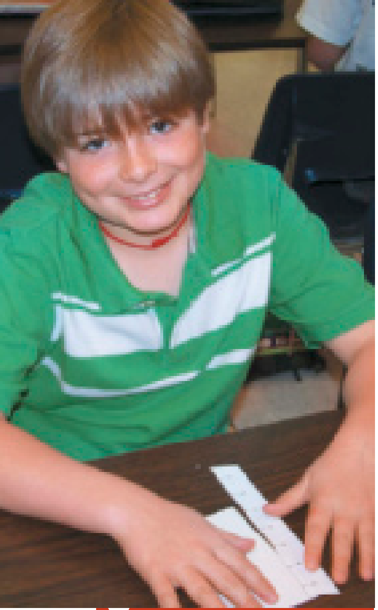
3. Can be debated

Third, students must be able to debate the issue. If, for example, the disagreement comes in the form of two solutions that contradict (or seem to contradict) each other, students should view the situation as problematic. This should lead them to choose a side of the disagreement and formulate a mathematical argument to prove their viewpoint. The situation could also offer students a chance to weigh both sides if they are unsure of which side is correct.

Students took either side of the argument in the ten hundreds versus one thousand disagreement. Some students were unsure of which side was correct. Although a few students straddled the issue, unable to decide on a position to take, every student was able to actively participate in the debate through argumentation or self-reflection.

The teacher's role

Teachers play a key role in the success of engaging students in mathematical disagreements. Before the disagreements, the teacher must work to establish a classroom environment that values risk taking, open discussion, and debating ideas. Students must learn to value and respect one another's opinions to the point that *all* students feel comfortable contributing to the discussions. If this environment has not been set up, disagreements can be negative, frustrating



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The proper classroom environment is key to building students' thinking and their participation, as well as avoiding negative disagreements.

experiences for students. If, on the other hand, students have become accustomed to analyzing student work and agreeing or disagreeing with discussion points, then disagreements provide opportunities to build students' thinking.

During the disagreement, the teacher has a two-fold responsibility in facilitating the discussion. First, when they initially engage in disagreements, students will have the inclination to share their thoughts and opinions without really considering the reasoning expressed by their classmates. The sample dialogues we have provided do not demonstrate this tendency, as they were taken later in the school year. We did have to address these tendencies, however, so that students would begin to actively listen to one another. To facilitate this change in behavior, we used questions such as these: Can you repeat what Sarah just said? Do you agree with what Marcus said? How is what Marcus said different from what Sarah said?

The second aspect of teacher responsibility is to fight the urge to participate during the discussion. Although refraining from sharing what you think—or from telling students what the correct answer is—can be extremely difficult, teachers risk losing valuable opportunities for learning to occur. Instead of thinking deeply about what has been said, sharing their reasoning, making sense of what others say, or revealing their understandings and misunderstandings, students will look to the teacher to resolve the disagreement and assume that the teacher is correct.

After the disagreement, the teacher must decide how to proceed:

- **What if** students are able to come to a mathematically correct consensus? Then the teacher must follow up with a question or task that uses that knowledge, allowing the issue to arise again, if necessary.
- **What if** students cannot come to a consensus? Then the teacher must design a follow-up task that will provide them with a deeper understanding of the mathematics necessary for resolving the disagreement.
- **What if** they all agree on an idea—and the idea is wrong? Then the teacher must design a follow-up task that will force students to confront their misunderstanding.

Regardless of whether students reach a correct consensus or disagree with one another, following up on this work is key to developing the type of mathematical understandings that we want our students to obtain.

Conclusion

Is 4×8 the same as 8×4, or are they different? Is it possible for the long side of an index card to measure both five-and-a-half inches *and* six inches? Do ten flats represent ten hundred or one thousand? As teachers, we most likely use writing prompts, tasks, and questions similar to those presented here. But we do not always recognize the potential these strategies have for evoking mathematical disagreements. As the previous examples demonstrate, creating opportunities for disagreement offers students the chance to engage in reasoning and sense making. By establishing the appropriate classroom environment and ensuring that all engage in the discussions, teachers encourage students to develop mathematical understanding and to thereby benefit from mathematical disagreements.

Specific questions that lead to disagreement

Consider questions (followed by justification) that will most likely lead to a mathematical debate among your students:

1. Is this equation true? 25 = 25

Students who believe this is a false statement will assert that you cannot have just a 25 on the left-hand side of the equation. For more information on this operational view of the equal sign, see Carpenter, Franke, and Levi (2003).

2. Look at the first triangle. Is it the same triangle if I turn it?

Original triangle   Turned triangle

Students who base their decision on how the triangle looks will say the original triangle has one horizontal side and two diagonal sides, whereas the turned triangle has three diagonal sides. For more information, see Clements and Sarama (2000).

With each of these questions, it may be the case that no disagreement occurs as all students supply the correct answer and support it with mathematically accurate justifications. If, however, such a question caused a disagreement with your students in past classes, it will most likely lead to a similar debate with your current students.

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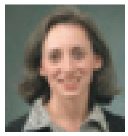

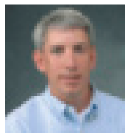

Angela T. Barlow, abarlow@olemiss.edu, is an associate professor of mathematics education at the University of Mississippi in Oxford. She is interested in developing teacher beliefs that align with standards-based instruction. **Michael R. McCrory**, **mrmccrory@gmail.com**, is a faculty member at Victory University. He is interested in the enhancement of pedagogy through self-reflection on teaching practices, personal beliefs, and external influences.