Catalyzing Change in High School Mathematics
Public Review Document

Preface

The last three decades have seen significant progress in the teaching and learning of mathematics in the United States. The current standards-based reform era was initiated by the National Council of Teachers of Mathematics (NCTM) with the publication of *Curriculum and Evaluation Standards for School Mathematics* in 1989. NCTM's original Standards and subsequent Standards publications, including *Principles and Standards for School Mathematics* (NCTM 2000), *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM 2006), and *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009), served as the impetus for the various mathematics and statistics standards produced in the United States over three decades.

The current discourse surrounding mathematics education and standards centers on students' preparation in mathematics and statistics for postsecondary education or a career. These goals are clearly important and will remain so, but as this publication makes clear, the preparation envisioned by these standards is not all that is important. Students should also leave high school with the quantitative literacy and critical thinking processes needed to make wise decisions in their personal lives. They should be able to determine whether or not truth claims made in scientific, economic, social, or political arenas are valid. They should have an appreciation for the beauty and usefulness of mathematics and statistics. And they should see themselves as capable lifelong learners and doers of mathematics and statistics. Never have the broader aims of mathematics and statistics education been more important than they are today when truth itself at times seems elusive.

Ever more rigorous standards, along with increased knowledge and implementation of research-informed instructional practices summarized in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014) have contributed to near record-high achievement and a positive long-term trend since 1990 in mathematics learning, as measured by the National Assessment of Educational Progress (NAEP), at both the elementary and the middle school levels (NCES 2015).

The steady improvement in mathematics learning seen at the elementary and middle levels has not been shared to the same degree at the high school level. Despite the increase in the percentage of high school students enrolling in upper-level mathematics courses over the last three decades (Dossey, McCrone, and Halvorsen 2016), high school NAEP scores have remained essentially flat for more than a decade (NCES 2015), and fewer than 50% of U.S. high school graduates in 2016 were considered ready for college-level mathematics work, as measured by their ACT mathematics scores (ACT 2016). Clearly, there is room to improve the mathematics learning experiences and outcomes of high school students. Equally clear is the fact that the current status of high school mathematics learning should not be blamed on high school mathematics teachers. The system of high school mathematics is complex, and it is the system and its structures that need to be examined and improved.

Because the system of high school mathematics education is so complex, changes to that system necessitate the engagement of all stakeholders involved in it. Therefore, *Catalyzing Change in High School Mathematics* is written for classroom teachers; counselors, coaches, specialists, and leaders; school, district, and state administrators; curriculum developers; and policymakers at all levels with the goal of beginning a serious discussion of the issues that *Catalyzing Change in High School Mathematics* outlines.

Catalyzing Change identifies and addresses some of the challenges to making high school mathematics and statistics work for each and every student. The purpose of *Catalyzing Change* is fivefold:

- Explicitly broaden the purposes for teaching high school mathematics beyond a focus on college and career readiness.

- Catalyze a serious discussion of the challenges facing high school mathematics as well as recommendations for implementing the actions necessary to overcome those challenges.

- Define imperatives for high school mathematics in the areas of structures, instructional practices, curriculum, and pathways for students.

- Identify essential concepts for focus that all high school students should learn and understand at a deep level in an equitable common mathematics pathway shared among all students.

- Provide examples of pathways that include 2½ years of mathematical study expected of high school students, followed by 1½ years of alternate paths of study, differentiated by postsecondary education and career goals.

It is critical to view this publication as merely the beginning—the catalyst for a serious and sustained effort on multiple levels to engage all who have a stake in the system of mathematics education of high school students in a coordinated, collaborative effort to improve the learning experiences and outcomes of each and every high school student. We owe this effort not only to our students but also to ourselves as we work together to create and nurture the society we wish to inhabit.

Matt Larson

President, 2016-2018

National Council of Teachers of Mathematics

Introduction

Although high school mathematics in the United States has changed in some ways over the last few decades, those changes have been relatively small compared with the changes that society has undergone in the same period. Although the content and emphases in high school mathematics courses have changed over the last century, for most students the high school mathematics courses they are offered remain Algebra 1, Geometry, and Algebra 2. First recommended by the Committee of Ten in 1892, the pathway of Algebra 1, Geometry, and Algebra 2 is still the primary route offered by the majority of high schools to students in the United States.

The Need to Catalyze Change

Today mathematics is at the heart of most innovations in the information economy. Mathematics serves as the foundation for STEM careers and, increasingly, careers outside STEM, and mathematical and statistical literacy are needed more than ever to filter, understand, and act on the enormous amount of data and information that we encounter every day. The digital age inundates us with numbers in the form of data, rates, quantities, and averages, and this fact of 21st-century life increases the importance of and need for our students to be mathematically and statistically literate consumers, if not producers, of information.

Quantitative skills are now demanded by disciplines such as the humanities and social sciences, where technical competence was not previously important. And increasingly, those prepared in mathematics and statistics are afforded more opportunities for professional advancement. Although it is impossible to predict the jobs of tomorrow (Wolfe 2013), it is clear that the 21st-century workplace will put a premium on students who are able to leverage a deep understanding of mathematics together with an ability to apply their understanding and procedural fluency to answer questions and solve problems that they have never seen before or may not exist today (NRC 2012).

For decades, a clear and consistent call for change has sounded in various reports, including *An Agenda for Action (*NCTM 1980), *A Nation at Risk* (National Commission on Excellence in Education 1983), *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), *Principles and Standards for School Mathematics* (NCTM 2000), and *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009). Despite the progress that the mathematics education community has made to improve mathematics instruction and learning in K–grade 8, an implementation gap persists between the calls for change and the comprehensive actions needed to support all high school students to learn and appreciate mathematics and statistics, to prepare them sufficiently for postsecondary education opportunities or a career (particularly in STEM), and to equip them with the quantitative skills necessary to make sound decisions in their lives and as members of our democratic society.

The Mathematical Association of America (MAA) has declared that the status quo with respect to undergraduate mathematics education is “unacceptable” (Saxe and Braddy 2015). The evidence suggests that the status quo with respect to high school mathematics learning outcomes is similarly unacceptable. Long-term trend NAEP assessment results at grade 12 have been flat for nearly 50 years, and, more concerning, learning opportunities are inequitable (Dossey, McCrone and Halvorsen 2016). In addition, U.S. high school students’ achievement has been sliding relative to that of their international peers, as measured by TIMSS Advanced and PISA (Mullis et al. 2016; OECD 2016b). When the potential of so many students is squandered, the loss is not only to individual students but also to society at large.

Focusing High School Mathematics

Catalyzing Change in High School Mathematics is a response to the need expressed by members of the mathematics education community for NCTM to bring focus to the high school mathematics and statistics standards (Larson 2016). Dominant national mathematics standards at the high school level lack the focus of mathematics standards at the K–8 level (Heiten 2015), and this shortcoming presents challenges to both teachers and students. Simply put, in most schools it is challenging for high school teachers to teach at the desired level of rigor—to develop students’ conceptual understanding, procedural fluency, and problem-solving skills while cultivating positive student mathematical identities, given the sheer number of standards that teachers are expected to teach and students are expected to learn. To support high school mathematics teachers, a focus on the essential concepts from algebra, geometry, probability, and statistics is critical if all students are to learn and understand foundational mathematics at a deep level. Knowledge of these essential concepts is necessary to open up professional and personal opportunities, as well as to cultivate a rich set of tools that students can use to apply mathematics meaningfully to understand and critique the world they inhabit.

Catalyzing Discussion and Improvement

Providing focus in high school mathematics, however, will not be enough. Given the complex system of high school mathematics education, collective action on the part of a variety of stakeholders is necessary to effect real change and improvement that will remove the structural barriers that impede mathematics teachers and learners of mathematics alike. Therefore, another goal of *Catalyzing Change in High School Mathematics* is to engage the multiple communities that influence and are responsible for high school mathematics and statistics education in a serious conversation leading to actions on the structures necessary to support educators in an effort to improve students' classroom experiences. These communities include, but are not limited to, state education policymakers, high school administrators and school boards, high school counselors, high school mathematics and statistics teachers, middle school administrators, middle school counselors, middle school math teachers, curriculum developers, and postsecondary educators.

Catalyzing Change identifies existing challenges in high school mathematics and statistics and indicates directions for improvement. These include the following:

- Creating equitable structures in high school mathematics—confronting the impact of student and teacher tracking and support systems.
- Supporting equitable instructional practices—transforming the teaching and learning experience for high school mathematics to support each and every student.
- Outlining what this publication calls Essential Concepts for Focus—key concepts that all high school students need as a foundation, regardless of postsecondary education and career goals.
- Developing equitable and common pathways—providing examples of an equitable and common high school experience that features alternative paths of study for all students.

The conversation will not be easy, since the challenges are real and longstanding. Making a difference will require collaboration, communication, and work across diverse groups and communities. It is our collective responsibility to the students of today and tomorrow to engage in these discussions and work to catalyze positive improvement in the experiences and outcomes in high school mathematics for each and every one of our students.

The Purposes of School Mathematics

The question that students most frequently ask in a high school mathematics classroom is likely to be, “Why do I have to learn this?” Many teachers of high school mathematics have at one time or another struggled to answer this question to the satisfaction of students. Clearly, the response, “It is necessary for the next course,” or “It is necessary for graduation,” is insufficient.

In recent decades, policymakers have emphasized usefulness for college and career readiness as the purpose of high school mathematics—a purpose that is clearly important. The obvious value of college and career readiness is one of the reasons why many states and school districts over the last decade have increased their high school graduation requirements in mathematics (Dossey, McCrone, and Halvorsen 2016). However, college and career readiness should not be, and is not, the only purpose of high school mathematics.

The purposes of high school mathematics are not fundamentally different from the purposes of school mathematics across K–12 or education in general, but they need to be considered specifically with respect to high school mathematics education. These purposes empower students to—

- expand professional opportunities;
- understand and critique the world; and
- experience wonder, joy, and beauty.

One difference between students' experiences of these purposes in K–grade 8 and their sense of them in high school mathematics is that K–8 mathematics is more visible in students' daily lives than is the mathematics that they study in high school. In K–8 mathematics, numbers and geometry in the curriculum are more directly evident in the world—students find examples in their daily needs for counting, sharing, using money, building, and drawing—and they find evidence of them in many news stories. However, the algebra and geometry that students study in the high school curriculum are less evident in the world that they encounter in their daily lives. As a result, many students question the need to study high school mathematics beyond the fact that it is required for graduation or for entrance to a postsecondary education program.

The multiple reasons for teaching and learning mathematics should be clear to both teachers and students as a powerful way to increase student engagement with and motivation to learn mathematics.

Expanding Professional Opportunity

The current standards-based reform effort has focused on the purpose of high school mathematics education for future education and employment—college and career readiness—or what mathematics education philosopher Paul Ernest labeled "necessary mathematics" (Ernest 2010). Beyond personal professional opportunity, mathematics education in the United States, particularly since World War II, has emphasized the importance of mathematics education for national economic and defense interests (Tate 2013), and this in turn has given mathematics status as a subject in the high school curriculum. A limitation of this narrow purpose for mathematics education is that it positions mathematics simply as something that “students need” (Gutiérrez 2012).

Unquestionably, a strong high school mathematics education opens doors to expanded professional opportunities, and this purpose of high school mathematics is a real expectation of students and parents alike (D’Ambrosio 2012). Although high school mathematics has long been seen as necessary for entry to a number of postsecondary education programs that lead to expanded career options, today the role of high school mathematics as essential for access to careers in science, technology, engineering and mathematics (STEM) is receiving increased attention from policymakers and business leaders.

This increased attention to preparation for STEM careers is in part due to data that indicate that growth in the number of STEM jobs that the economy generates over the next decade will accelerate, particularly compared with the numbers of jobs in other professions (Vilorio 2014). In 2012, the President’s Council of Advisors on Science and Technology (PCAST) reported that the United States needs a million more STEM graduates over the next decade to meet the demands in STEM and STEM-related careers (PCAST 2012). Additional data indicate that beginning salaries and salary growth for STEM majors will outpace those of other college majors and in other careers (NACE 2016). Clearly, professional opportunities are available to students in STEM, and high school mathematics is necessary preparation for access to these STEM and STEM-related careers.

Understanding and Critiquing the World

Mathematics is deeply embedded in many aspects of daily life, both seen and unseen. These include modern communication systems, transportation systems, medicine, manufacturing, finance, security, science, engineering, technology, and the increasing use of "big data" sets to make decisions (NRC 2011).

Students should be able to identify, interpret, evaluate, and critique the mathematics embedded in social, scientific, commercial, and political systems, as well as the claims made in the private and public sectors and in public interest-group pronouncements (Ernest 2010). For example, high school mathematics can help students develop precision in reasoning so that they do not deduce conclusions from false statements, use inductive reasoning inappropriately, or mistakenly conclude that the inverse of a true statement is also true.

High school mathematics can help prepare students to investigate the world, understand it, and critique it. When high school mathematics courses integrate tasks that, for example, address income distributions, ecological issues, health rates, taxing structures, or lending practices, students have "access to rich, rigorous mathematics that offers opportunities and self-empowerment for them to understand and use mathematics in their world" (Stinson and Wager 2012, p. 10). When high school mathematics is approached as a tool to give students a better understanding of their role as members of our democratic society, students are more likely to actively engage in their communities, and are more likely to appreciate their potential power to challenge injustices and contribute to societal improvement (Gutstein and Peterson 2013).

Experiencing Wonder, Joy, and Beauty

High school mathematics can potentially cultivate in students a sense of wonder, beauty, and joy, and this is an important, but often neglected purpose for teaching mathematics. The ability to reflect and experience awe is a distinctly human activity, and seeing the world through a mathematical lens can help students experience wonder and beauty in the world in unexpected places. A student with a deep understanding of mathematics can connect counting strategies to Pascal's triangle or appreciate the role of Fibonacci numbers in nature. The fact that many phenomena of the world can be explained mathematically is itself wondrous. Einstein famously asked, "How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?" Mathematical ideas are a triumph of human creativity, and framing mathematics in this way can motivate students to unleash their own creative potential.

Mathematics has been shaped by the cultures in which it was developed, and it is important to see the role of mathematics in history and society (Ernest 2010). Great ideas of mathematics are as beautiful as great works of art, and just as in the study of art, students can learn to see mathematics as expressions of beauty fashioned by drama and struggle. High school content from algebra or calculus can help students see the beauty of describing the physical world with mathematical precision and help them appreciate the role of mathematics in building modern civilization.

Mathematics has a cultural heritage and history that spans every part of the globe. High school mathematics should strive to highlight the contributions of a variety of cultures to mathematics—not just Western contributions. High school mathematics can illustrate the diversity of ways in which mathematics has been developed and appreciated by many cultures—the Chinese knew about the relationship between the sides and hypotenuse of a right triangle thousands of years before the Pythagoreans. Valuing and developing a better understanding of the multiple contributions that various cultures have made to mathematics can cultivate and nurture positive student mathematical identities.

The Power of Multiple Purposes

Undeniably, preparing high school students in mathematics for college and for careers in important. However, addressing critical thinking and the beauty of mathematics is also important. When the multiple purposes of school mathematics are emphasized, students are prepared to “flourish as human beings,” no matter what paths they take in life or what profession they choose (Su 2017). A multi-purposed high school mathematics curriculum plays a critical role in the cultivation of students who become fully engaged members of democratic society, who contribute to society in positive ways, and who become human beings capable of achieving their full potential, personally and professionally, through the intellectual experiences of their mathematics education.

Creating Equitable Structures

The challenge is clear: “We have a long-standing, thoroughly documented, and seemingly intractable problem in mathematics education: inequity” (Aguirre at al. 2017, p. 125). Each and every student’s access to a high-quality mathematics education is critical, given the importance of mathematics to individuals who are able to flourish fully in their personal and professional lives.

In *Principles and Standards for School Mathematics* (NCTM 2000), NCTM first offered the Equity Principle: “Excellence in mathematics education requires equity—high expectations and strong support for all students” (p. 12). In *Principles to Actions* (NCTM 2014), the Council reinforced its commitment to equity in the Access and Equity Principle, which states, “An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential” (p. 59). This publication adopts Gutiérrez’s perspective that equity in mathematics education will not be achieved until it is no longer possible “to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language” (Gutiérrez 2002a, p. 153).

Despite this continued focus on equity, the mathematics education community has yet to be able to address the structural barriers, including teacher and student tracking, which continue to contribute to inequitable student outcomes (Martin 2015). The creation and adoption of more rigorous mathematics standards in most states over the previous decade have provided an impetus for change and improvement. However, barriers still exist, and these are often political, cultural, and systemic. Current reform efforts that focus largely and almost exclusively on standards, with some attention to improved instructional practice, are unlikely to address equity concerns without also addressing and dismantling the conditions, institutional policies and practices, and structures that stand as barriers to the creation of positive mathematical experiences for students—particularly those who are not experiencing success in mathematics. Addressing these issues will require action on the part of nearly all stakeholders in high school mathematics education: high school teachers, school and district administrators, policymakers at the district and state levels, and teacher educators and researchers in higher education.

Many of these systemic barriers prevent students from developing a sense that learning mathematics is possible for them, and the barriers produce opportunity gaps that lead to unacceptable differentials in learning outcomes. Although a variety of factors beyond the classroom affect students' opportunity to learn mathematics (Stiff and Johnson 2011), it is necessary for all stakeholders in education to seriously rethink their current practices and revise their policies to better serve all students by removing the structural barriers that deny students access to a high-quality mathematics experience. Three significant structural barriers are directly within educators' sphere of influence: student tracking, teacher tracking, and the instructional support offered to students, both before and during high school.

Distinguishing between Tracking and Acceleration

Catalyzing Change in High School Mathematics draws a distinction between tracking and acceleration. Tracking is defined here as the placement of students into qualitatively different and, in some cases, terminal mathematics course progressions or pathways—courses or pathways that do not prepare students for any continued study of fundamental mathematical concepts. Often placement into these tracks is based on a variety of nonacademic reasons (Stiff and Johnson 2011), such as perceived academic achievement, race, socioeconomic status, gender, language, or other expectations ascribed to students by adults.

Tracking should be distinguished from acceleration. Acceleration, as described in the NCTM Position Statement *Providing Opportunities for Students with Exceptional Mathematical Promise* (NCTM 2016), may be appropriate if a student has demonstrated deep understanding of grade-level or course-based mathematics standards significantly beyond his or her current level.

The NCTM Position Statement is clear that “when considering opportunities for acceleration in mathematics, care must be taken to ensure that opportunities are available to each and every prepared student and that no critical concepts are rushed or skipped.” The statement’s emphasis on ensuring that “no critical concepts are rushed or skipped” is too often overlooked. Some school districts push to skip key concepts in a “race to calculus.” Such a race is often misguided. Simply put, mathematics learning is not a race, and evidence suggests that students who speed through content without developing deep understanding are the very ones who tend to drop out of mathematics when they have the chance (Boaler 2016, p. 192).

A recent study (Bressoud, Mesa, and Rasmussen 2015) from the Mathematical Association of America (MAA) addressed the issue of acceleration to calculus in K–12 schools. Bressoud (2015), recommended against racing to calculus and suggested instead providing “an alternative to calculus in high school that focuses on strengthening students’ understanding of algebra, geometry, trigonometry, and functional relations while building problem solving skills.” Similarly, a joint position statement of MAA and NCTM states that the “ultimate goal of the K–12 mathematics curriculum should not be to get students into and through a course in calculus by twelfth grade but to have established the mathematical foundation (and disposition toward mathematical work) that will enable students to pursue whatever course of study interests them when they get to college” (MAA and NCTM 2012).

Appropriate acceleration is and must be acceleration along a common pathway that provides each student with an opportunity to learn the same essential concepts but that eventually branches to a limited set of fluid course progressions that are open to all students. The common pathway presented in *Catalyzing Change* is specifically designed to ensure that each student acquires a common focused mathematical foundation. Any mathematical pathway that supports equitable outcomes must be characterized by the learning of significant mathematics that ensures that future student learning options remain open.

Acceleration is often entangled with issues of equity and privilege (Grissom and Redding 2016). According to the NCTM Position Statement, acceleration “opportunities must be open to a wide range of students who express a higher degree of interest in mathematics, not just to those who are identified through traditional assessment instruments” (NCTM 2016). Traditionally, acceleration decisions have tended to be based on student performance on traditional assessments, which have privileged some students and failed to meet the needs of others—often those who lack resources or support.

As the NCTM Position Statement also notes, “Exceptional mathematical promise is not a fixed trait; rather, it is fluid, dynamic, and can grow and be developed … exceptional mathematical promise is evenly distributed across geographic, demographic, and economic boundaries” (NCTM 2016). If the demographics of students accelerated in mathematics in a school or district are not evenly distributed across racial, linguistic, cultural, and economic boundaries, then reflection and analysis are called for to determine why not, and actions should be taken to remove whatever biases or structural barriers lead to this inequitable outcome.

Student Tracking

Tracking students into qualitatively different course progressions, where some students have access to mathematics instruction that prepares them for postsecondary education opportunities while others do not, reinforces the misguided notion that only some people—stereotypically, middle-class and white students—are capable of achieving in mathematics (Boaler 2011).

Tracking in mathematics instruction in the United States is a significant issue. Research has long documented the negative consequences to students placed in the “low” track (Oakes 1985). Despite the known negative consequences, research indicates that students from marginalized groups continue to be tracked in ways that offer them less access to highly qualified mathematics teachers and less access to college preparatory pathways in mathematics (Nasir 2016). A recent report from the Organisation for Economic Cooperation and Development (OECD) found that more than 70% of students attend schools where the principal reports that students are grouped by “ability” for mathematics instruction (OECD 2016a). A report from the Brown Center on American Education that examined data covering two decades found that, beginning in eighth grade, three-fourths of students were tracked in mathematics courses (Loveless, 2013).

The learning opportunities provided to students in these different tracks are hierarchically different (Boaler, Wiliam, and Brown 2000). Students in the privileged “top” track experience mathematics instruction that cultivates their mathematical identities, conceptual understanding, and critical problem-solving and thinking skills. Students placed in the “low” track tend to focus on rote procedures, with instruction devoting little or no attention to developing their understanding or their belief that mathematics is something that they can do. As a result, students in the low track do not receive the high-quality mathematics education that all students deserve. The replication of this experience year after year has long-term negative effects on students' learning outcomes and their mathematical identities (Stiff and Johnson 2011).

Flores (2007) argues that these low expectations all too often become self-fulfilling prophecies. Once students are placed in a low track, moving out of that track is very difficult (Stiff and Johnson 2011). The opportunity gap created by tracking students into weak, low-quality, and dead-end course progressions puts students further and further behind in the curriculum and leads to what is commonly referred to as the “achievement gap.” This gap is more accurately labeled as the opportunity gap, since it is largely manufactured in schools by tracking practices that place students into low-level courses and dead-end learning pathways, often with less effective instruction (Flores 2007). The tracking of students into instructionally qualitatively different and dead-end course progressions is essentially “educide,” since it severely limits, and all too often ends, students' opportunities in mathematics and mathematics-related careers.

In recent years, many high schools, at least on paper, have done away with traditional three-track sorting (gifted and talented, regular, and remedial). However, in some cases tracking now exists in new forms at the high school level. For example, algebra course offerings may sort students into one- or two-year versions of the same course. Students in the two-year version are essentially denied the opportunity to learn as much mathematics as their peers who are placed in the one-year course (AERA 2006). In other new forms of tracking, some schools may label courses as Algebra or Algebra II, although they are in no way rigorous enough to merit these course titles. Different groups of students are then tracked into these different versions of algebra (Stein et al. 2011). Again, the result of these tracking practices is the same: inequitable learning outcomes.

The research is unequivocal: the mathematics experience of students placed in lower tracks with less access to rigorous curriculum and high-quality instruction is qualitatively different from the mathematics experience of students not placed in lower-level tracks. Furthermore, this difference has long-term negative effects on achievement and affective outcomes for the students in lower tracks and ultimately exacerbates learning differentials (Oakes et al. 1990; Schmidt 2009; Schmidt, Cogan, and McKnight 2011; Stiff and Johnson 2011; Tate and Rousseau 2002). *Catalyzing Change in High School Mathematics* recommends an equitable and common pathway focusing on essential concepts in all content domains for the first 2½ years, which, in addition to providing a cohesive high school mathematics curriculum, ensures the highest-quality mathematics education for all students. Students must not be tracked into dead-end course progressions that do not expand their options for future learning but instead limit their professional and personal opportunities and damage their mathematical identities and sense of agency (defined in the next section, Implementing Equitable Instruction).

In addition, *Catalyzing Change* recommends that students enroll in meaningful mathematics courses all 4 years that they are in high school. By enrolling in meaningful mathematics courses in every year of high school, students will have opportunities to explore important mathematics in high school beyond the concepts that populate the equitable common pathway and will not experience a gap in their mathematics learning. A one- or two-year gap in high school mathematics enrollment can make it challenging for students to reengage in postsecondary learning of mathematics if their educational and professional plans change. Evidence suggests that students with 4 years of high school mathematics score significantly higher on college entrance exams and require less remediation in college (Achieve 2013). Furthermore, mathematics that builds on the shared essential concepts provides high school students who will not immediately pursue postsecondary courses with important knowledge and skills for different careers.

De-tracking students in high school mathematics can be one of the most challenging policy changes to enact. Although tracking may be considered the “default” method for organizing mathematics in high school (Oakes 2008), ample evidence indicates that de-tracking leads to success for more students (Boaler 2002; Boaler and Staples 2014; Burris and Weiner 2005; Strutchens, Quander, and Gutiérrez 2011). Evidence also suggests that successfully de-tracked high school mathematics programs share, among other factors, two important characteristics: (1) the connections and meaning in mathematics are emphasized by teachers, and (2) the curriculum is focused on key mathematical ideas (Horn 2006). Implementing a high school mathematics curriculum focused on essential concepts and approaching those essential concepts with rigor, as recommended in *Catalyzing Change*, can increase the likelihood of successfully de-tracking traditional high school mathematics programs.

Teacher Tracking

Mathematics teachers themselves are often tracked, with the most experienced teachers, or those who are perceived to be the most competent, assigned to upper-level mathematics courses, while inexperienced teachers are assigned to entry-level-mathematics courses (Darling-Hammond 2007; Strutchens, Quander, and Gutiérrez 2011). In a study of 29 districts in 16 states, marginalized students had access to less effective instruction than non-marginalized students, and that lack of access to high-quality instruction persisted over time (Isenberg et al. 2013). Educators have a professional and moral obligation to carefully examine whether “teacher assignments and tracking practices are helping or hindering equity” (Lubienski 2007).

Catalyzing Change in High School Mathematics recommends that high school mathematics teachers in the same department have “balanced” teaching assignments. This means that each mathematics teacher’s assignment should include both upper-level and entry-level mathematics courses (Gutiérrez 2002b). Balancing teaching assignments deepens teachers’ knowledge of the overall curriculum expectations, can reduce burnout among new teachers, can populate collaborative teams with experienced teachers, and can develop among teachers a collective sense of responsibility for all students (Strutchens, Quander, and Gutiérrez 2011).

Su (2017) argues that "we have to recognize that even if people are just, even if they desire to be just, a society may not be just if its structures and practices are not also just." The practices of student and teacher tracking in mathematics education are longstanding barriers to offering each and every student access to a high-quality mathematics education. Tracking practices are not just, and they contribute to unjust differentials in student learning outcomes. Student and teacher tracking practices must be dismantled if we are to achieve the goal of supporting each and every student in reaching his or her potential in mathematics.

Supporting Student Success in a Common Pathway

Rigorous K–8 mathematics standards such as the Common Core State Standards for Mathematics (NGA and CCSSO 2010) as well as those adopted by most states, coupled with research-informed effective teaching practices (NCTM 2014), should provide students entering high school with the mathematics foundation necessary to succeed in a common pathway beginning in ninth grade. As previously noted, although there has been significant progress overall in mathematics learning in K–grade 8 over the last 30 years, it would be naïve to assume that every student matriculates in high school with the necessary mathematical background for immediate success in a common pathway. The reasons for this are numerous, but one reason is the existence of K–8 structures that prevent student acquisition of the necessary mathematical foundation.

One such structure includes student tracking at the K–8 level. Although tracking becomes more obvious in high school, it is not just a high school concern (Flores 2008). Tracking frequently starts much earlier in K–12 education and often becomes visible only when students reach high school. Loveless (2013) reported that in 2011 nearly two-thirds of fourth-grade teachers reported using “ability grouping” in math instruction. All too often this practice begins in the primary grades. For example, in some schools, if there are three first-grade teachers, at the end of kindergarten the students are rank-ordered and divided into three groups, with one first-grade teacher receiving the highest-achieving one-third of the students, the second teacher receiving the middle one-third, and the third teacher receiving the final one-third. Such practices do a disservice to students.

A second obstacle is the traditional way in which many elementary- and middle-level math interventions are structured. Elementary- and middle-level interventions in mathematics, if they exist, frequently remove students from the grade-level curriculum. Although this may support students in acquiring skills and concepts that they missed previously, they may continue to fall behind in the grade-level curriculum if they are removed from it, and they are unlikely to be fully prepared to enter a common pathway without additional support.

Effective interventions provide *additional* instructional time instead of removing students from grade-level instruction (Baker, Gersten, and Lee 2002). This additional time often takes the form of a second period of mathematics instruction at the middle level. To maximize effectiveness, this additional time should be fluid, allowing students to enter and leave as needed and determined by the results of frequent classroom-based formative assessments (Larson and Andrews 2015). Additional targeted instructional time is a support strategy used by some high-performing countries (Barber and Mourshed 2007; OECD 2011).

Effective targeted instructional support should be focused on content that is connected to and promotes the grade-level curriculum (Balfanz, MacIver, and Byrnes 2006; Burris, Heubert, and Levin 2006) and not simply a review of low-level procedural skills. Some scholars have recommended this type of additional instructional time, with students receiving tailored instruction during one period to support success in their core mathematics course, as an effective strategy to support English language (EL) students (Thompson 2017).

At the high school level, double-period versions of courses in a common pathway can also be an effective approach to support students who may experience extraordinary challenge with the content. For example, evidence indicates that double-period versions of algebra courses (and geometry and second-year algebra) can be an effective way to support students in being successful in algebra (Cortes, Goodman, and Nomi 2015;

Larson and Andrews 2015). Double-period versions of courses are preferable to two-year versions of courses, since two-year courses often put students further behind and limit their ability to complete a high school mathematics pathway that will prepare them for postsecondary opportunities (Education Trust 2005).

Finally, to support all students' success in a common pathway, teachers need to collaborate on instructional issues in professional learning communities. In these settings, they can expand their pedagogical strategies to engage each and every student in meaningful mathematics learning and develop the shared expectation that all students will be successful in a rigorous course progression (NCTM 2014; Strutchens, Quander, and Gutiérrez 2011). Necessary support for teacher collaboration is described more fully in the next section.

Implementing Equitable Instruction

The ways in which students experience mathematics have a significant impact on the way in which they identify themselves as doers of mathematics. Implementing equitable instruction begins with the recognition that students develop mathematics Identities that are highly contextualized and mediated by environments.

Mathematics Identity

In mathematics classrooms, mathematics identity is mediated by how students engage with mathematics. Consequently, mathematics teaching involves not only helping students learn concepts and develop skills and understanding but also empowering students to see themselves as capable of participating in and being doers of mathematics. In too many high school mathematics classrooms, students are passively engaged with mathematics, and little mathematical discourse occurs. Reasoning and sense making are rarely encouraged, and mathematics is positioned as having little relevance to students' lives or experiences. In other classrooms, students are active participants, engaging in reasoning and sense making, striving to make their mathematical thinking visible to others, using multiple forms of discourse, and critiquing their world through the use of mathematics.

Aguirre, Mayfield-Ingram, and Martin (2013) defined mathematics identity “as the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives” (p. 14). Depending on the context, one’s mathematics identity reflects a sense of oneself as a competent performer who is able to do mathematics. How students are positioned to participate in mathematics affects not only what they learn but also how they come to see themselves as learners. The ways in which students view themselves as learners of mathematics greatly influence how they participate (Nasir and Hand 2006). Developing students’ identities should be part of teachers’ daily work, in which they use teaching practices that focus on mathematics, leverage multiple mathematical competencies, affirm mathematical identities, challenge marginality, and draw on multiple resources of knowledge (Aguirre, Mayfield-Ingram, and Martin 2013).

Mathematics Agency

The ways in which students participate in mathematics and express their mathematics identities determines their level of agency. Agency refers to the expression of one’s identity (Murrell 2007). Students tell others through words and actions who they are and what their purpose is in a particular setting, space, and situation. In mathematics classrooms, agency is expressed in the ways that students engage in productive struggle, take risks to make their mathematical thinking visible, and understand that learning results when they successfully leverage an approach that works for them. A high sense of agency allows and encourages students to continue with a rigorous course of study in mathematics.

Mathematical agency is about participating in mathematics in ways that are meaningful, both personally and socially (Berry 2016). Because participation in mathematics involves problem solving, reasoning, sense making, discourse, modeling, and appropriate use of tools, effective learning settings implement core equitable mathematics teaching practices that support mathematical agency (Bartell et al. 2017). Equitable mathematics teaching practices support identity and agency by creating structures for having students' mathematical ideas considered during instruction, supporting students in viewing themselves as having ownership of mathematical meaning and in coordinating enterprises across contexts to strengthen this ownership (Aguirre, Mayfield-Ingram, and Martin, 2013; Oppland-Cordell and Martin 2014).

Equitable Mathematics Teaching Practices

Teachers can build students' mathematics identity when they use teaching practices effectively to position students as being mathematically competent by creating opportunities for them to demonstrate agency and efficacy (Wenger 1998). The eight Mathematics Teaching Practices articulated in *Principles to Actions* (NCTM 2014) provide a framework for making connections between high-leverage teaching practices that support the development of identity, agency, and competence. These eight Mathematics Teaching Practices are described below with connections from each practice to identity and agency. Each description ends with a suggestion of the connection.

Establish mathematics goals to focus learning. Mathematics goals have two components: they describe the mathematical concepts, ideas, or methods that students will understand as a result of instruction, and they identify the mathematical practices that students will be learning (NCTM 2014). Establishing norms for participation involves creating structures to position each and every student as participatory and recognizing that participation builds agency (Turner et al. 2013). Within this teaching practice, teachers must establish classroom norms for participation that position each and every student as a competent mathematics thinker.

Implement tasks the promote reasoning and problem solving. Effective use of mathematics tasks motivates learning and helps students build new mathematical knowledge through problem solving (NCTM 2014). Tasks that require reasoning, problem solving, and modeling (i.e., tasks with high-cognitive-demand) result in a positive orientation toward mathematics and oneself as a doer of mathematics (Boaler and Staples 2008). Within this teaching practice, teachers must use tasks in ways that develop positive dispositions toward mathematics and build students' mathematical identity.

Use and connect mathematical representations. Mathematical representations are of particular importance in helping students to advance their understanding of mathematical concepts and procedures, make sense of problems, and engage in mathematical discourse (NCTM 2014). The use of multiple representations allows students to draw on multiple sources of knowledge (Boston et al. 2017). Drawing on multiple sources of knowledge acknowledges the mathematical, social, and cultural resources that students bring to mathematics. Teachers who use this teaching practice effectively validate the resources that students bring to mathematics and connect instruction to students' experiences and interests.

Facilitate meaningful mathematical discourse. Discourse gives students opportunities to share ideas and clarify understanding, construct mathematical arguments, develop a language to express mathematical ideas, and learn the mathematical perspectives of others (NCTM 2014). Through discourse, students realize that their work and thinking serve an important role in the mathematics classroom, thus positioning themselves and others as mathematically competent and reducing hierarchical status in mathematics classrooms (Boston et al. 2017). In implementing this teaching practice, teachers create structures that position students as mathematically competent and capable of engaging in sharing their mathematical thinking, connecting with peers to understand others' mathematical ideas, and participating in mathematical arguments.

Pose purposeful questions. Purposeful questioning encourages students to explain and reflect on their own and others' thinking; it allows teachers to discern what students know and understand and to use these insights to adapt lessons to meet the needs of students (NCTM 2014). Students who are often asked questions requiring them to explain their reasoning may be positioned differently from students who are primarily asked questions not requiring explanation. The types of questions that students are asked can support positive mathematical identity and agency by positioning students as thinkers and doers of mathematics (Aguirre, Mayfield-Ingram, and Martin 2013). Teachers implementing this teaching practice pose purposeful questions and are mindful of which students they are asking which types of questions and whose ideas are privileged during discourse.

Build procedural fluency from conceptual understanding. Conceptual understanding and procedural fluency are critical and connected components in the development of mathematical proficiency (NCTM 2014). Mathematics instruction that focuses solely on remembering and applying procedures advantages students who are strong in memorization skills and disadvantages student who are not (Boston et al. 2017). Consequently, focusing primarily on the memorization of procedures may convey the message that mathematics is not about knowing and doing but is about memorizing. By contrast, teachers who make a practice of building fluency from conceptual understanding routinely connect conceptual understanding with procedural fluency so that students can make meaning of the mathematics and develop a positive disposition toward mathematics.

Support productive struggle in learning mathematics. Teaching that embraces productive struggle provides opportunities for students to delve deeply into relationships among mathematical ideas and to develop understanding that leads them to apply their learning to new problem situations (NCTM 2014). This teaching practice involves giving students time to grapple with mathematical ideas (Hiebert and Grouws 2007). Grappling provides opportunities for students to develop a sense of agency by taking ownership of their mathematical thinking. Working within this teaching practice, teachers allow time for students to engage with mathematical ideas and provide supports through purposeful questioning to support perseverance and identity development.

Elicit and use evidence of student thinking. Eliciting and using students' ideas require that teachers attend to more than just whether an answer is right or wrong. This teaching practice requires focusing on common patterns of reasoning and attending to how students understand a task and how ideas are developed over time (NCTM 2014). Whose ideas are elicited and used in the classroom has strong implications for mathematical identity and agency. Eliciting mathematical ideas from students who are perceived as always giving the right answer positions correctness as more valuable than mathematical thinking. Consequently, students may not share their thinking and may participate only when they believe that they may have a correct answer. By contrast, teachers who make a practice of eliciting and using evidence of students' mathematical thinking position each and every student as mathematically competent.

Figure 1 provides a crosswalk between the eight Mathematics Teaching Practices in *Principles to Actions* (NCTM 2014) and equitable mathematics teaching practices. The equitable mathematics teaching practices focus on practices supporting identity and agency. The recommendations in the crosswalk are not intended to be exhaustive; rather the intent is to provoke ideas and serve as first step for teachers who are intentional about implementing equitable teaching practices. Practices highlighted below draw on the work of Boston and colleagues in *Taking Action: Implementing Effective Teaching Practices in Grades 9-12* (2017) and other researchers in mathematics education.

Mathematics Teaching Practices: Supporting Equitable Mathematics Teaching	
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.	● Establish learning progressions that build students' mathematical understanding, increase their confidence, and support their mathematical identities as doers of mathematics. ● Establish high expectations to ensure that each and every student has the opportunity to meet the mathematical goals. ● Establish classroom norms for participation that position each and every student as a competent mathematics thinker. ● Establish classrooms that promote learning mathematics as just, equitable, and inclusive.
Implement tasks the promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points	● Engage students in tasks that provide multiple pathways for success and that require reasoning, problem solving, and modeling, thus enhancing mathematics identity and sense of agency. ● Engage students in tasks that are culturally relevant.

and varied solution strategies.	• Engage students in tasks that allow them to draw on their funds of knowledge.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.	• Use multiple representations so that students draw on multiple resources of knowledge to position them as competent. • Use multiple representations to draw on knowledge and experiences related to the resources that students bring to mathematics (culture, contexts, and experiences). • Use multiple representations to promote the creation and discussion of unique mathematical representations to position students as mathematically competent.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and	• Use discourse to elicit students' ideas and strategies and create space for students to interact with peers to value multiple contributions and diminish hierarchal status among students (i.e., perceptions of

comparing student approaches and arguments.	differences in smartness and ability to participate). • Use discourse to attend to ways in which students' position one another as capable or not capable of doing mathematics. • Make discourse an expected and natural part of mathematical thinking and reasoning, providing students with the space and confidence to ask questions that enhance their own mathematical learning. • Use discourse as a means to disrupt structures and language that marginalize students.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.	• Pose purposeful questions and then listen to and understand students' thinking to signal to students that their thinking is valued and makes sense. • Pose purposeful questions to assign competence to students. Verbally mark students' ideas as interesting or identify an important aspect of students strategies to position them as competent.

	• Be mindful of the fact that the questions that a teacher asks a student and how the teacher follows up on the student's response can support the student's development of a positive mathematical identity and sense of agency as a thinker and doer of mathematics.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.	• Connect conceptual understanding with procedural fluency to help students make meaning of the mathematics and develop a positive disposition toward mathematics • Connect conceptual understanding with procedural fluency to reduce mathematical anxiety and position students as mathematical knowers and doers. • Connect conceptual understanding with procedural fluency to provide students with a wider range of options for entering a task and building mathematical meaning.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently	• Allow time for students to engage with mathematical ideas to support perseverance and identity development.

provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.	● Hold high expectations, while offering just enough support and scaffolding to facilitate student progress on challenging work, to communicate caring and confidence in students.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.	● Elicit student thinking and make use of it during a lesson to send positive messages about students' mathematical identities. ● Make student thinking public, and then choose to elevate a student to a more prominent position in the discussion by identifying his or her idea as worth exploring, to cultivate a positive mathematics identity. ● Promote a classroom culture in which mistakes and errors are viewed as important reasoning opportunities to encourage a wider range of students to engage in mathematical discussions with their peers and the teacher.

Fig. 1. Crosswalk: Eight Mathematics Teaching Practices (NCTM 2014) matched with equitable mathematics teaching practices

Supporting Effective Mathematics Teaching Practices

Providing high school students with more rigorous classroom instruction requires improvements in teaching practices (Boston and Smith 2009). Improvements in teaching practices must not only consider ways to make mathematics content more accessible to students but also support students in seeing themselves as knowers and doers of mathematics. Such improvements in teaching practices must consider mathematics identity and agency as essential constructs for raising achievement levels and mathematical dispositions for each and every student. Too many high school students struggle through mathematics classes, experiencing repeated failure, and often they disengage from mathematics, finding little intellectual challenge as they are asked only to memorize and execute routine procedures (Boaler and Staples 2008). Not enough students are offered opportunities to connect different mathematical ideas, apply methods to different situations, and use mathematical modeling. Improvements in high school mathematics teaching depend on improving the collective and individual professional knowledge, and effectiveness of each and every teacher (NCTM 2014).

NCTM's Professionalism Principle states:

> In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for

> personal and collective professional growth toward effective teaching and learning of mathematics (NCTM, 2014; p. 99).

For teachers, a key aspect of the Professionalism Principle is recognizing that their own learning is never finished and that they must build a culture of professional collaboration that is driven by a sense of interdependence and collective responsibility (Berry and Berry 2017). Too many high school mathematics teachers plan their lessons alone, teach behind closed doors, keep ideas and activities to themselves, infrequently observe their colleagues' teaching, and rarely review videos of their own teaching (Leinwand 2015).

Time and professional isolation are two obstacles that have direct impact on the professional collaboration needed to improve high school mathematics teaching practices (Berry and Berry 2016). The lack of time to collaborate among teachers contributes to professional isolation. For many teachers, the regular school day does not provide adequate time or opportunity to engage in the collaboration necessary to improve teaching practices. Professional isolation leads to inconsistencies in teaching practices, creating inequities in student learning (Feiman-Nemser, 2012). Isolation allows "good" and "bad" teaching to go unnoticed or noticed silently. The research provides compelling indications that teachers who collaborate inside and outside their school will work together to notice and interpret aspects of effective teaching, will share a collective responsibility for student learning, and will work to ensure equitable distribution of resources (Boaler and Staples 2008; Horn 2005; van Es and Sherin 2002).

Time for collaboration must be embedded in the daily work of teachers. Although scheduling challenges exist, schools can purposefully create spaces for informal and formal interactions of teacher collaboration. Schools and teachers should not focus only on what is expected to be taught but also consider what strategies are effective and equitable for teaching and how they will know that students have learned (Kanold and Larson 2015). Collaboration moves teaching from a solitary experience to a public space where teachers engage in critical conservations about teaching and learning.

A strong body of research suggests that equitable mathematics teaching practices in secondary schools have positively impacts on students' participation and academic success (Boaler and Staples 2008; Larnell, Bullock, and Jett 2016; Gresalfi et al. 2009; Gutstein 2016; Gutiérrez 1996; Hand 2012). When teachers implement practices that support students in engaging in discourse, decision making, exploration, and strategizing, students develop positive mathematics identities and agency. The research suggests that equitable mathematics teaching practices are inclusive when they acknowledge that students bring knowledge and resources from their communities and make community-based knowledge and resources an integral part of mathematics teaching.

Essential Concepts for Focus in High School Mathematics

The content domains of high school mathematics include algebra and functions, statistics and probability, and geometry. The concepts and mathematical practices that students engage in while learning the content from these domains not only form the foundation for the study of mathematics at a higher level, but provide fundamental knowledge and skills required for the study of other disciplines, such as the sciences, economics, and the social sciences, where mathematical modeling, statistical inference, and the analysis of big data are increasingly used, as well as the development of the critical-thinking and problem-solving skills needed for the workplace and for successful participation in democratic society.

Supporting Fundamental Knowledge and Skills

The high school mathematics curriculum should support students in knowing *how* to solve problems (procedural fluency), understanding *why* procedures work (conceptual understanding), and knowing *when and how* to use mathematics (problem solving and application), all while building in students a positive mathematics identity and sense of agency. In addition, the high school mathematics curriculum should help students to see the big picture, key concepts, and connections between the concepts that they are learning. This includes connections between and among the major content domains, among various mathematical representations, and between contexts that relate mathematics to other subjects and to the real world. Helping students to see these connections builds a deeper and more lasting understanding and a deeper appreciation for the value of mathematics as a problem-solving tool (NCTM 2000).

Technology is driving changes that should be reflected in the high school mathematics curriculum. As a result of advances in technology, “some mathematics is now more important, some mathematics is less important … and some mathematics is possible for the first time” (Seeley 2006). Digital technology can serve three main functionalities (Drijvers, Boon, and Van Reeuwijk 2010): (1) as a tool for doing mathematics and statistics, (2) as a learning environment for fostering the development of conceptual understanding, and (3) as a learning environment for practicing skills. Dynamic technologies can help students transfer mental images of concepts to visual interactive representations that can lead to a better and more robust understanding of the concept. Computer algebra systems or their equivalent can be tools to explore patterns, develop generalizations, and carry out mathematical processes. Students should have opportunities to use dynamic interactive technology in all content domains to explore and deepen understanding of concepts, to interpret mathematical representations, and to employ complex manipulations necessary to solve problems. Mastery of skills should not be a prerequisite for using the technology in any content area, but instead the focus should be on understanding and interpreting the results (Roschelle et al. 2000; Sacristan et al. 2010).

As stated earlier, one of the purposes of *Catalyzing Change in High School Mathematics* is to identify a set of Essential Concepts for Focus that all high school students should learn at a deep level of understanding. These essential concepts represent the most critical content from the content domains—the deep understandings that are important for students to remember long after they have forgotten how to carry out specific techniques or apply particular formulas. However, learning mathematics involves more than just acquiring content and carrying out procedures. The processes and practices that students engage in as they deepen their understanding of key mathematical concepts are as important as students' acquisition of the key concepts themselves.

Developing Mathematical "Habits of Mind"

Mathematical modes of thought, sometimes referred to as habits of mind, "are useful for reasoning about the world from a quantitative or spatial perspective and for reasoning about the mathematical content itself, both within and across mathematical fields" (Levasseur and Cuoco 2003, p. 27).

Numerous scholars and documents have offered lists of mathematical habits of mind, including the Standards for Mathematical Practice in the Common Core State Standards for Mathematics (NGA and CCSSO 2010), the Process Standards from *Principles and Standards for School Mathematics* (NCTM 2000), and the reasoning habits in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009). Some of the mathematical habits of mind that are in common to these various lists include the following (NGA and CCSSO 2010; Levasseur and Cuoco 2003; Cuoco, Goldenberg, and Mark 2010; Seaman and Szydlik 2007).:

- Making and testing conjectures
- Generating a variety of approaches to solving a problem
- Using prior knowledge to guide problem solving
- Creating strategic examples and non-examples to see underlying structure
- Looking for patterns
- Thinking algorithmically
- Using alternative representations
- Classifying carefully
- Using precision in language
- Modeling with mathematics
- Reasoning and proving

These habits of mind, and others, are developed as students engage in challenging mathematical tasks and the problem-solving process. The importance of these mathematical habits of mind lies in their contribution to students' ability to understand and critique the world. They are consistent with the analytic and interpretative skills needed in today's workplace (NRC 2012). These mathematical habits support the purposes of learning mathematics outlined in *Catalyzing Change*: they expand professional opportunity, help students understand the world, and increase the likelihood that students will experience wonder and joy when engaging both in mathematical study and in mathematics in the world.

Two of these habits, proof and modeling, represent broader cross-cutting approaches and provide opportunities for teachers to enhance essential content, as well as to employ each of the eight Mathematics Teaching Practices (NCTM 2014) and their connections to equitable teaching. These habits are described below and highlighted within each content domain.

Proof. The idea that truth can be established by using reasoning and proof is a standard of knowledge that is quite special to mathematics. Unlike other fields, where new knowledge may invalidate old knowledge, true statements in mathematics are not overturned by new knowledge. A geometry fact or an algebraic identity or the validity of a statistical method remains true for all time, and even the reliability of a statistical claim has a provable mathematical quantification of uncertainty.

Mathematical knowledge often grows in a cycle of inquiry and justification. As Bass (2015) describes, knowledge often progresses through a trajectory marked by exploration, discovery, conjecture, proof, and certification. The first three phases involve the reasoning of inquiry, and the last two involve the reasoning of justification. Exploration and experimentation are activities in which students discover patterns and form, reject, and/or refine conjectures. Discovery and conjecture often involve inductive reasoning—making inferences from specific examples to general statements—whereas proof involves determining when and why a conjecture does or does not hold by using deductive reasoning—applying general theorems or statistical methods to specific instances. Together, these activities form a cycle because the proof process may lead to rejecting and refining conjectures. It may also help students to generalize discoveries and develop understandings to help them remember an important claim and how to apply it.

It is important that students see the process of establishing mathematical and statistical truths as the entire cycle of inquiry and justification. The inductive aspect of developing mathematical knowledge is just as important as the deductive aspect. For this reason, the entire cycle of inquiry and justification—both inductive and deductive approaches—should be integrated into each content domain.

Moreover, the cycle of inquiry and justification should be leveraged to implement the eight Mathematics Teaching Practices and their connections with equitable teaching. In the reasoning of inquiry, teachers can ask purposeful questions and facilitate discourse in ways that support mathematical identity and agency. In the reasoning of justification, teachers can elicit evidence of student thinking to build conceptual understanding.

Modeling. A mathematical model is a mathematical representation of some real-world process or phenomenon that is under examination, in an attempt to describe, explore, or understand it. Mathematical modeling is the creative, often collaborative, process of developing these representations. Modeling always requires decision making, which involves determining which aspects of the phenomenon to include in the model and which to suppress or ignore and what kind of mathematical representation to use. As noted by mathematician Henry Pollak, throughout the modeling process, both the real-world situation and the mathematics must be taken seriously (Pollak, 2012).

The mathematical modeling cycle (SIAM 2016) requires students to formulate a problem or question; state assumptions and define variables; apply concepts to the problem or question; analyze and assess the model and possible solutions; iterate, refine, or extend the model; and report results. The “messiness” of authentic modeling problems is also a critical part of the process. In mathematical modeling, students will develop techniques, use tools, and employ different perspectives as they apply concepts to create models based on authentic problems drawn for their own lives or from situations likely to be encountered later in life.

Even though modeling is universally presented as a cycle, modeling in the classroom takes many different forms. Although it is important that students have some opportunities to develop extended models, it is not typically productive to expect students to experience the full cycle as an introduction to modeling. Students can develop a sense of modeling and become comfortable with some of the tools and techniques of mathematical modeling by engaging in small, simple, one- and two-step problems. Small modeling activities both build students’ confidence in their mathematical decision making and give them opportunities to develop skills in collaboration with others. The sharing of ideas enables each student to experience new perspectives on the topics and gain new insights, since they get to see mathematical ideas through another student's eyes. Participating in modeling also requires extensive communication, further enhancing student learning. Building on experiences with small modeling activities, students are able to benefit from participation in the complete modeling cycle.

Modeling, both as small activities and through the full cycle, should be used within each of the content domains to investigate how different phenomena or events might perform or behave under given constraints or assumptions. Such models provide opportunities to predict what might, could, or probably will happen and, as a consequence, can empower students as decision makers in informed and productive ways.

The process of mathematical modeling provides opportunities for teachers to employ each of the eight Mathematics Teaching Practices and their connections with equitable teaching. Indeed, a desirable outcome of mathematical modeling is that as teachers implement effective practices through modeling, students begin to adopt those practices as well: posing purposeful questions, connecting various representations, and enjoying productive struggle (NCTM 2014).

Mathematical modeling is central and essential to providing high school students with the knowledge, skills, and dispositions needed to make greater sense of the world. Modeling mathematics and statistics should be key components of any high school mathematics program.

Essential Concepts for Focus

The next sections of *Catalyzing Change* describe Essential Concepts for Focus in each of the key content domains and address why they are important in the context of the foundation established for the continued study of mathematics and its use in a variety of careers. The essential concepts are designed to populate the learning targets of an equitable common mathematics pathway for the first 2½ years of high school mathematics. Students who acquire an understanding of these concepts will have continued mathematical opportunities open to them, both in high school and after high school. The general introduction to each content domain section provides an overview of the "big picture" and the value of that content domain for the high school curriculum, providing insight into the question, "Why do we need to teach this?" or, alternatively, "Why do I need to learn this?" The roles of technology, proof, and modeling, as well as connections between the content areas, are highlighted. Examples illustrating the Essential Concepts for Focus are presented explicitly and embedded in More4U throughout each content domain section.

The Essential Concepts for Focus were determined on the basis of a review of recommendations made by several key documents and organizations, including the Common Core State Standards for School Mathematics *(*NGA and CCSSO 2010*)* and related state-level standards, the GAISE and SET reports published by the American Statistical Association (Franklin et al. 2005, Franklin et al. 2015, GAISE College Report

2016), *Principles and Standards for High School Mathematics* (NCTM 2000), *Focus on High School Mathematics: Reasoning and Sense Making* (NCTM 2009), the GAIMME report published by COMAP and SIAM (2016), and *A Common Vision for Undergraduate Mathematical Sciences Programs in 2025* (Saxe and Braddy 2015).

Essential Concepts for Focus: Algebra and Functions

Some general themes in mathematics are the systematic study of patterns, the analysis of change, and the power of deduction. The study of algebra and functions is an important opportunity for students to see how mathematics addresses these themes and their applications to problems of life and society.

The study of algebra In high school is the systematic exploration of the arithmetic of expressions involving unknown or variable quantities as well as methods for finding those unknowns or determining relationships between variables. The English word *algebra* comes from an Arabic word, transliterated to English as *al-jabr*, that means “restoration” or “completion.” Persian mathematician Al-Khwarizmi used the word in a 9th-century book in which he systematically solved general quadratic equations, and *al-jabr* referred to the operation of moving a subtracted expression from one side of an equation to become an added expression on the other side. Today, we use the word *algebra* much more broadly to describe a collection of procedures that enable one to work with unknown or variable quantities, reasoning logically to determine their values or reach further conclusions about them.

The study of algebra is inextricably linked with the study of functions, which are fundamental objects in mathematics that model many situations in life, often those that involve change. *Functions are descriptions of a dependent relationship between one quantity and another.* How does the incidence of a disease depend on age? How does the lift of an airplane wing depend on the velocity of the plane? Whereas the ordered pair (*x, y*) learned in algebra can represent a single point on a graph or a solution to an equation, a function $y = f(x)$ represents the relationship between a whole set of ordered pairs and can be treated as a single entity with characteristics and behaviors. Functions are sometimes described by algebraic expressions, but they may also be specified graphically or inferred from data. The notion of a function is broad and flexible and can apply to many situations, including some that are not algebraic or quantitative. In high school, understanding and working with functions typically requires the algebraic knowledge and skills for working with equations and expressions, but it also requires a qualitative understanding of properties of that dependence. This understanding can be enhanced by the use of technological tools to visualize graphs of functions and explore their characteristics and behaviors.

Algebra and functions are the language of generalization: they are useful for describing general patterns of relationships by means of a formula and for answering all questions of a particular type at once rather than solving many questions individually (Usiskin 1995, 2014). For example, it is helpful to have a general formula to calculate the interest earned in a bank account, given the interest rate and time period, rather than solving a specific problem in each case from first principles. Moreover, classifying the structure of the expressions that arise (e.g. linear, exponential) provides a deeper understanding of the dependences between quantities. How does the population growth depend on time? A facility with algebra and functions can be used to classify the relationship ("exponential") and bring knowledge of one problem to bear on many others. Although the essential concepts for functions seek to develop proficiency with linear, exponential, and quadratic functions, students should also have experiences with graphical representations of a variety of other functions to increase their understanding of the broad applicability of functions.

In high school, students use the tools that they have developed for manipulating expressions from earlier mathematics courses to work with more complicated expressions and to connect their understanding of algebra with functions and their graphs. Algebraic knowledge, skills, and understandings are further developed in high school as students connect mathematical representations with their contexts and explore functional relationships such as periodic functions (e.g., temperature graphs) from a broad perspective, often using technology to produce graphs of functions from applied situations. A large part of the study of algebra and functions in high school mathematics should be devoted to applying algebra and functions in the process of mathematical modeling—using functions and algebraic expressions to describe relationships between quantities in context, make predictions about the outcome in new situations, and use data to refine their models—using contexts from the real world.

In contrast to the abstraction of high school mathematics, elementary mathematics focuses on the study of number. However, number remains important at the high school level. In high school, all students study the irrational numbers, together with the rational numbers, integrated within the content domains. Many extensions of number—for example, complex numbers, vectors, and matrices—are studied in high school by some students but are beyond the set of essential concepts needed as a foundation by each and every student for postsecondary and career readiness. As mathematical applications increase in importance and complexity in the curriculum, numbers in context become even more important and often represent quantities with attached units. As a consequence, students need to attend to units as a way to understand problem situations and mathematical models as well as using units as a guide to solution strategies and the evaluation of conclusions. Choosing appropriate scales in graphing quantities and deciding appropriate levels of accuracy in approximating values are also important in solving problems within and outside mathematics.

Section 1: Algebra

Recognizing and using structure in expressions is a central element of algebraic reasoning in high school. Many situations have a common underlying structure (such as a linear or a quadratic relationship between two variables), while others involve expressions that have a recognizable structural composition, such as the sum of a constant and a product, or the product of a sum and a difference. Rewriting expressions in equivalent forms can make different characteristics or features of the expression visible; for example, factored expressions can be analyzed to see when the value of the expression will be positive, negative, or zero. When expressions are used in describing contexts or connected with other representations, parts of expressions, such as terms, factors and coefficients, can be interpreted in terms of the context or representation. Graphs and tables of values can be used to reason that two expressions are equivalent, but definitions and properties of operations are necessary to provide a mathematical justification. Technology can be used to produce different forms of quadratic or exponential expressions, and students should be able to interpret each representation in relation to a context.

With the assistance of technology, students can use a slider to change the powers in situations such as those in figure 1 to reinforce the difference in a sum to a power and a product to a power, helping them build a mental image of how the results differ. Or they can drag points on a number line, as illustrated in figure 2, and consider questions such as the following:

- “When, if ever, will the two expressions be equal? How do you know?”
- “For what values of *a* and *b* will the differences in the two expressions be positive? Negative? How can you tell?”
- “How would your answers change if the exponent was two? Make a conjecture and then change the exponent and see if your thinking was accurate.”

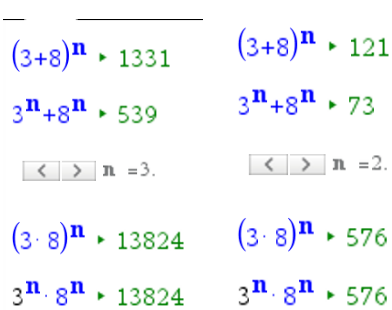


Fig. 1. Powers and sums

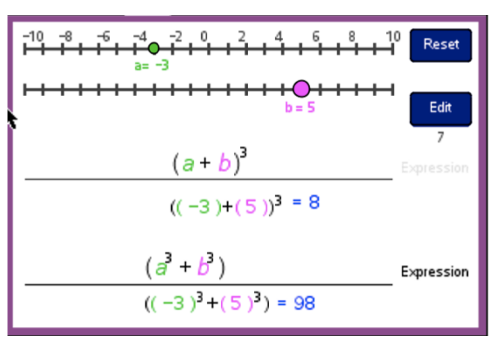


Fig. 2. Generating expressions

Solving equations and inequalities is a creative process, often with multiple paths for finding a solution. The structure of an equation or inequality (including one-variable linear and quadratic equations, inequalities, and systems of linear equations in two variables) can be purposefully analyzed to determine an efficient strategy to find a solution, if one exists. Students should focus on justifying their solution path by appealing to mathematical structure and context, recognizing the special cases when an equation is an identity, and when a solution does not exist for a particular domain. Some equations cannot be solved by following a set of procedures but only by reasoning about the numbers and the structure of the equation. For example, reasoning about exponents and the zeros of the quadratic will lead to the solutions for $(x^2 - 5x + 5)^{(x - 1)} = 1$.

Expressions, equations, and inequalities can be connected with contexts both within and outside mathematics. Constraints arising from the mathematics or a context can be represented by equations and inequalities. Values satisfying the constraints are solutions to equations and inequalities, and can be interpreted in terms of the original context and questions. Becoming proficient in using technology to put in sliders and observing what changes and what remains invariant can help students understand and interpret the effect of varying assumptions or conditions, thus becoming purposeful users of mathematical notation.

Students in algebra should engage in the modeling cycle to solve problems such as maximizing profit for a small business, balancing the cost of a product and corresponding effect on demand, or organizing a bank of elevators for the morning rush so that everyone can arrive on time without having to take the stairs or arrive earlier. In the process, students have to state assumptions and define variables (e.g., original cost, number of elevators, number of stories); apply algebraic concepts to the problem or question; create a model of the situation and use it to find possible solutions. Opportunities to compare solutions across the class can lead students to iterations and improved models as well as understanding how algebra can be used to solve “messy” and complex problems.

Proof is essential in algebra and should not be reserved only for geometry. Proving conclusions in algebra often relies on reasoning deductively from definitions and statements of relationships that are accepted as true, such as the distributive property of multiplication over addition. Asserting that something is true in algebra means assembling a proof that justifies the thinking leading to the statement by using these properties and clearly stated definitions. For example, students might be asked to prove or provide logical arguments for the assertion that if 15 students write down the day of the week on which they were born, at least one day will be used three times. Or they might be asked to show that the number of possible handshakes between n people is $n(n - 1)/2$.

Essential Concepts of Algebra for Focus in a Common Pathway:

- **Expressions can be rewritten in equivalent forms by using the properties of addition, multiplication, and exponentiation to make different characteristics or features visible. For expressions describing contexts or connected with other representations, each part can be interpreted in relation to the context or representation, including multiple parts representing a single entity.**

- **Multi-term or complex expressions can represent a single quantity and can be substituted for that quantity in another expression, equation, or inequality; doing so can be useful when rewriting expressions and solving equations, inequalities, or systems of equations or inequalities.**

- **The solution to an equation or inequality in one variable or to a system of equations or inequalities in two variables, if one exists, are the values of the variable, or variables, that make the equation, inequality, or system true. Candidate solutions, whether generated analytically or graphically, must be checked in the original equation, inequality, or system of equations or inequalities to ensure that they are actually solutions.**

- **The structure of an equation or inequality (including one-variable linear and quadratic equations, inequalities, and systems of linear equations in two variables) can be purposefully analyzed to determine an efficient strategy to find a solution, if one exists. Many equations and systems of equations**

(or inequalities) can be solved by creating equations (or inequalities) that are simpler to solve.

- **Expressions, equations, and inequalities can be connected with contexts, both within and outside mathematics—in particular, contexts related to linear, quadratic, and exponential situations.**

Section 2: Connecting Algebra to Functions

The ability to recognize and apply the distinction between equations and functions in interpreting, constructing, and solving problems is important for continued study in high school and beyond (Carlson, Madison, and West 2010). Relationships between two variables in algebra are generally about a single ordered pair that satisfies the conditions defining a particular relationship, whereas a function describes a set of ordered pairs that can be treated as an object with its own attributes (e.g., the function f is increasing for $x > 2$). A robust understanding of functions is essential for reasoning about and modeling situations—the very situations that allow students to use mathematics to understand and critique the world.

Functions may be described by using a variety of representations (e.g., tabular, verbal, analytical, graphical, recursive). Recognizing the dependent relationship between the input and the output of a function is necessary for thinking about how change in the input affects the output; this thinking allows for more global thinking about the function beyond a focus on discrete points. Studying the graphs of functions and connecting key features of graphs with the structure of algebraic expressions, as well as recognizing the roles of different parameters in changing some aspects of graphs while preserving others, are essential aspects of high school mathematics.

As high school students explore contexts in science and statistics, they recognize that graphs can be useful tools for answering a question or making a prediction. Creating graphs from data involves generating ordered pairs, selecting axis labels and scales, and, finally, creating a rule or equation that summarizes the relationship between the variables and relating that rule back to the original contexts, questions, and predictions. However, creating equations from graphs is also an important strategy that students often put into practice when given the results of a science experiment or observing behavior with respect to some independent variable such as time.

The coordinate plane provides a way to visualize relationships between quantities, and allows further understanding of connections between algebraic statements and the functions or relationships that they express. A graph associated with an equation consists of an infinite set of ordered pairs, each of which is a solution to the equation represented by the curve. Students explore how linear, quadratic, and exponential functions as well as other functions described graphically behave over intervals specified by describing the change in the output (dependent variable) for intervals of the independent variable, examining such features as rate of change, increasing, decreasing, maximum or minimum points, symmetry, and intercepts. These features of the graph can all be associated with appropriate algebraic representations for the functions.

Different symbolic forms of the same equation represent different contextual interpretations and connect with different features of the graph that describe the relationship between the variables. Consider for example, the different forms of a quadratic $y = a(x-h)^2 + k, y = ax^2 + bx + c,$ and $y = a(x-p)(x-q)$, in the context of the number of items sold, x, and the profit, y, in dollars for a particular company (Adapted from Illustrative Mathematics, www.illustrativemathematics.org/content-standards/HSA).

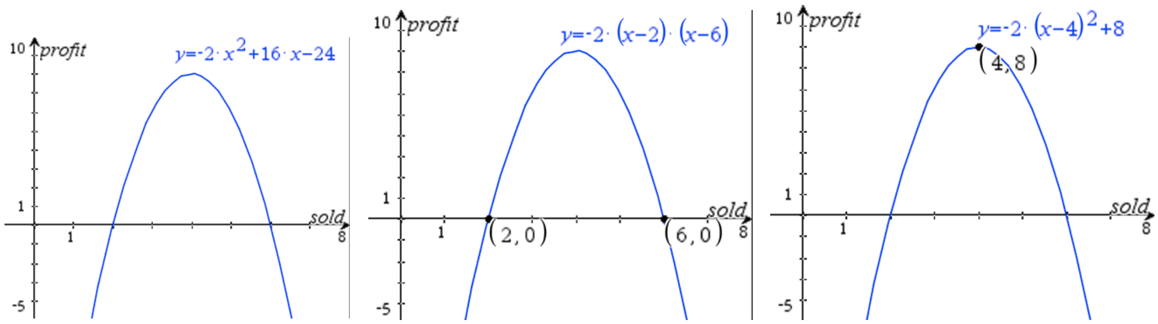
1. What do each of the following equations tell you in terms of the number of items sold and the profit earned?

$$y = -2x^2 + 8x - 24$$

$$y = -2(x-2)(x-6)$$

$$y = -2(x-4)^2 + 8$$

2. How do the features of the graph connect to the symbolic equation in each case?



Technology is useful both to do the work of mathematics (performing calculations, graphing, generating tables) and to develop conceptual images of important algebraic and mathematical concepts. As students use dynamically linked interactive technologies to explore concepts, to observe the impact on a graph as they change parameters, and to observe the impact on an equation as they drag points, they are able to focus their attention on the underlying mathematical structures and principles and are able to verbalize what the graphical or tabular representation is telling them instead of concentrating their efforts on the computation.

Essential Concepts Connecting Algebra to Functions for Focus in a Common Pathway:

- **Functions generalize the algebraic relationship in which equations in two variables represent an input and its corresponding output, often for a given rule or constraint. A function generalizes this relationship from a point-by-point relationship to an entity that consists of the entire set of ordered**

pairs as specified by the constraint, and this entity can be named and described in general terms.

- **Graphs can be used to approximate solutions of equations, inequalities, and systems of equations—in particular, systems of linear equations in two variables and systems of linear and quadratic equations (given or obtained by using technology).**

- **Linear, quadratic, and exponential functions can be represented graphically, and key features of the graphs, including zeros, intercepts, and, when relevant, rate of change of *y*, the dependent variable, with respect to a unit change in *x*, the independent variable, and maximum or minimum values, can be associated with and interpreted in terms of the algebraic representation.**

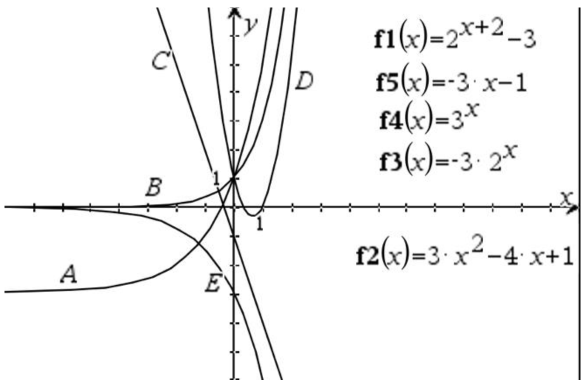
Section 3: Functions

Functions are a fundamental concept in mathematics. They are the basis of many mathematical applications and a mathematical tool to characterize the relationship between two sets in such a way that a set of inputs (the domain) maps to a set of outputs (the range), with each input mapped to exactly one output. To understand functions is to recognize that functions can be described by using a variety of representations (e.g., tabular, verbal, analytical, graphical, recursive) and that functions do not necessarily have to be quantitative in nature.

In high school, functions are often studied and understood as families of a particular “parent” function. Function families can be classified by examining characteristic behavior, such as end behavior, maxima and minima, intercepts, and symmetries. The ability to recognize and reason about particular attributes of functions is essential to understanding and using functions to model different phenomena appropriately and solve problems. Although it is important that students be able to recognize, construct, and apply attributes of exponential and quadratic functions, it is perhaps just as important that students see each of these families of functions in a more general sense, as a way to model and explain phenomena. When students compare and contrast different characteristics of functions, they develop a more complex, connected understanding of the structure of functions in general and structures that make families of functions unique.

The following example might be used to help students learn to describe functions in multiple ways and to become familiar with function notation.

Match each graph to a function expressed symbolically and to the verbal description. Explain your reasoning.



(i) The function is increasing for all values of x.
(ii) The function has an absolute minimum
(iii) The function is decreasing for all values of x.
(iv) The difference in $(f(x_1) - f(x_2))/(x_1 - x_2)$ is constant.

Rate of change, one defining characteristic of some function families, can capture how the input and the output of a function vary simultaneously. In high school, students continue to develop their understanding of the complex ideas of function and rate of change as they explore and come to know specific functions in greater depth. For example, students can discover that the rate of change for nonlinear functions is not constant, leading to a discussion of average rates of change over an interval. Just as students have previously experienced linear functions by using multiple representations, students should have frequent and varied opportunities to experience and reason about rate of change of families of functions by using multiple representations in various applied contexts.

Functions allow students to solve problems such as determining what speeds should be enforced to maximize traffic flow in a given area. Given a contextual situation, students can engage in the modeling process (SIAM 2016): formulate the problem; make assumptions and define variables; create functions to model the situation and analyze the model for possible solutions; iterate, refine or extend the model; and report results.

Essential Concepts of Functions for Focus in a Common Pathway:

- **Functions describe the dependent relationship between two sets in a way that maps inputs to outputs so that each input is mapped to exactly one output.**

- **Functions that are members of the same family have distinguishing attributes (structure) common to all functions within that family.**

- **Rate of change is a specific attribute of some functions that captures how the input and output vary together.**

- **Mathematical techniques involving algebra and functions can be used to model real situations involving making and changing assumptions, assigning variables, and finding solutions to contextual problems.**

Essential Concepts for Focus: Statistics and Probability

> *A major objective of statistics education is to help students develop statistical thinking. Statistical thinking, in large part, must deal with … [the] omnipresence of variability; statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in the data. It is this focus on variability in data that sets apart statistics from mathematics.* (Franklin et al., 2005, p. 6)

Statistics and probability concepts that are essential for all high school students include the ability to analyze data, to engage in the process of statistical inference, to understand conditional probability and independence, and to develop quantitative literacy to inform decision making. The concepts build on learning from earlier grades and provide a foundation for future study in more advanced high school courses, in postsecondary education, for problem solving in careers, and, most important, for the thinking and reasoning required of informed members of our democratic society.

All high school graduates will, as members of society, be presented with data-based claims in the course of their lives. Therefore, they must be able to examine these claims and become intelligent consumers of studies, capable of reasoning critically and asking questions about the implementation of the statistical process in those studies. This process includes the statistical question or questions of interest, the type of study design, the data collection process, the sources of variability in the data, how the data were analyzed, whether the conclusion is appropriate for the study, and whether the conclusion answers the original statistical question. *Stated simply, it is necessary that every high school graduate have the statistical reasoning skills to be a "healthy skeptic" when confronting statistical information.*

The digital revolution of the last few decades has made statistical thinking an even more compelling necessity in the modern world. Our actions now leave digital footprints, and these are driving the development of "smart" technologies that aim to make our lives more productive, as well as decisions (for good or bad) about what products will be marketed to us and what opportunities will be offered to us. The explosion of data has led to the emerging field of data science, a growing career option and a trend that demands the attention of every person who uses any smart technology. *All high school graduates need to have a solid understanding of statistics and probability so that they can be informed consumers as well as producers of innovation that depends on data science.*

Section 1: Data Analysis

Statistics is about data—numbers with a context (Cobb and Moore 1997). Data analysis includes "the examination of data for interesting patterns and striking deviations from those patterns" (Moore 1997, p. 3–4). The world today is awash with data that come in many different forms, and analyzing the data is crucial to nearly every aspect of society, including business, industry, health and human services, education, and government.

Investigating or analyzing these data implies posing relevant questions. A set of data is sometimes given as a fait accompli, and the user "interrogates" the data to uncover stories that they may contain. Other times, the data are collected in response to a question. The focus of data analysis is on the underlying structure of the data as a whole and how the analysis might point toward conjectures to explain why an event occurred, lay a foundation for predicting how events might unfold in the future, or be used to compare different subgroups.

At the high school level, the first step in analyzing data is to recognize that data come in two traditional forms, quantitative and categorical, and the techniques for analyzing the two forms are very different. For example, arithmetic means do not make sense for categorical data, and categorical data are typically organized in a table and represented in bar graphs or segmented bar graphs. Technology allows access to large data sets and can be used to organize and manage the data, take random samples, and test conjectures by resampling—all central elements in the emerging field of data science. The emphasis in data analysis is on the shape, center, and variability in a distribution. Variability is encapsulated in the concept of deviation, a foundational concept that appears in different forms in statistics.

In high school, students build from an understanding of the mean absolute deviation to develop an understanding of standard deviation, a measure of "typical" variability from the mean of a set of data, in which the deviations from the mean are squared (note that the sum of squared residuals will be a quadratic, which will always have a unique minimum as opposed to a sum of absolute values). Outliers (observations that deviate considerably from the basic structure of a distribution) raise important questions: is this an observation that could have occurred by chance, or is it possible that some other factor is involved? Boxplots are a convenient way to display outliers as well as very useful in comparing two or more distributions of quantitative data with respect to the same variable—an important skill in high school. However, a boxplot indicates little about the shape of a distribution, with the exception of revealing whether the distribution is approximately symmetric or asymmetric.

Associations between two categorical variables may be analyzed by organizing the data in two-way tables, interpreting the entries in the cells, and using segmented bar graphs to represent the data. Associations between two quantitative variables may be investigated by representing the paired data in a scatterplot and, when the pattern is linear, searching for a model that "fits" or summarizes the association. In choosing a model, statisticians make assumptions and consider the context as well the data themselves. Initial focus is on linear equations as a broad class of useful models that give an accessible entry to expressing and using mathematics to model the situation. If the underlying structural relationship indicates a linear trend, a line through the "center" of the cloud of points can summarize or capture the essential nature of the trend.

The process of fitting and interpreting models for possible associations between quantitative variables requires a high level of reasoning that involves insight, good judgment, and a careful look at a variety of options consistent with the context and with the questions posed in the investigation. Three important factors should be considered in analyzing the "goodness" of a fitted model: (1) minimizing the sum of squared residuals by using technology (the residual is the difference between the corresponding coordinate for a point on the line and the actual data value for a given explanatory variable), (2) analyzing the residuals for patterns (one should not be able to predict anything about the error in using the fitted line to predict; for example, the model will always overpredict for large values of the explanatory variable), and (3) considering the context itself. Under the assumption that a linear model is a relatively good fit, the slope (predicted rate of change) can be interpreted in the context of the data. In many situations, the explanatory variable is time, and such plots can be analyzed for trends or lack of trends over time.

Technology allows students to drag a line to generate various models to represent the relationship between two variables. Using technology, students can calculate different sums of the squared residuals as the line is manipulated, enabling students to search for a line that produces a minimum sum of these residuals as shown in figure 1. With this as a background, students can use the least squares regression feature of most graphing technology to choose the linear model that will actually minimize the sum of the squared residuals. They still need to pay attention and reason about the other "think abouts" for fitting a line. For example, figure 2 shows a regression line for the relationship between the distance of a property from "Go" in the game Monopoly and the price of a property and a representation of the squared residuals. Students might note that the costs for railroads and utilities are constant and thus are not necessary to consider in thinking about a potential model for the relationship between cost and distance from "Go." A careful examination of the residuals in figure 2 indicates that for properties close to “Go,” the linear model in the figure will overpredict the cost of the property, while for those farther from “Go,” the model will underpredict the cost. Along with the fact that the prices of railroads and utilities are constant, this suggests that another model might be more appropriate—in particular, one that disregards the railroads and utilities.

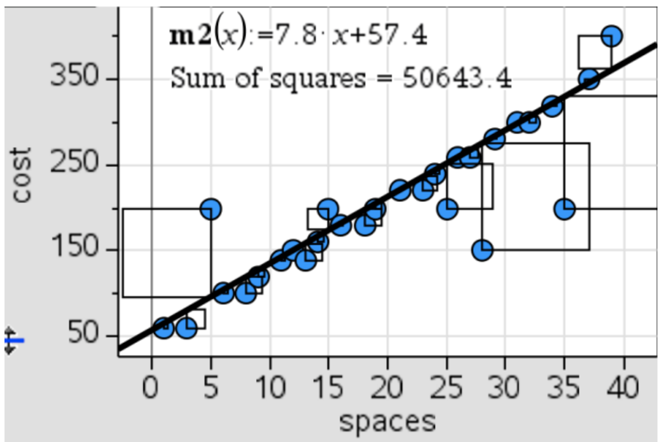


Fig. 1. Sum of squared residuals

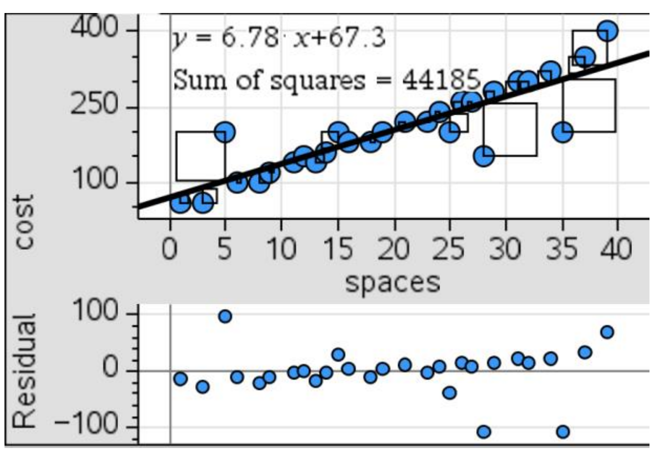


Fig. 2. Pattern in residuals

Under the assumption that a model is a relatively good fit, students should be able to interpret the slope (predicted rate of change) in the context of the data (e.g., 7.8 in figure 1 represents a predicted increase of $7.80 in the cost of a property for every 1 space further from "Go"). They should recognize that the $7.80 is an estimated rate of change.

Because data analysis techniques center on collecting and analyzing data, these techniques are invaluable in engaging students in the mathematical modeling cycle (SIAM 2016). Problems such as how to hire five employees across five departments with unequal workloads as fairly as possible or how to classify insects such as midges as those causing debilitating diseases and those that are harmless can be addressed by identifying variables, creating statistical models, and using these to examine the situation, changing assumptions and posing possible solutions. Regression techniques are also valuable tools in using functions to model "messy" or open-ended problems.

Essential Concepts of Data Analysis for Focus in a Common Pathway:

- **Data arise from a context and come in two traditional types: quantitative (continuous or discrete) and categorical. Technology can be used to "clean" and organize data, including very large data sets, into a useful and manageable structure—a first step in any analysis of data.**

- **Distributions of quantitative data (continuous or discrete) in one variable should be described in the context of the data with respect to what is**

typical (the shape, with appropriate measures of center and variability, including standard deviation) and what is not (outliers), and these characteristics should be used to compare two or more subgroups with respect to a variable.

- **The association between two categorical variables is typically described by using two-way tables and segmented bar graphs.**

- **Scatterplots, including plots over time, can reveal patterns, trends, clusters, and gaps that are useful in analyzing the association between two contextual variables.**

- **Analyzing the association between two quantitative variables should involve statistical procedures such as examining (with technology) the sum of squared deviations in fitting a linear model, analyzing residuals for patterns, generating a least squares regression line and finding a correlation coefficient, and differentiating between correlation and causation.**

- **Data analysis techniques can be used to develop models of contextual situations and to generate and evaluate possible solutions to real problems involving those contexts.**

Section 2: Statistical Inference

Members of society encounter studies reported in the media that contain generalizations about a population or the comparison of experimental treatment groups. These studies draw inferences beyond the analysis of the data distribution by quantifying, in context, the structure that emerges in sampling distributions for a statistic such as a sample mean or sample proportion. The statistical inference process builds on data analysis and is an important component in the development of quantitative literacy. The evolution in the studying of statistical inference from exploring and describing one sample to making inferences about a population is a big leap for students. As David Moore (1990, p. 127) stated, “The question of inference in simplest form is how to draw conclusions about a population parameter on the basis of statistics calculated from a sample.”

Experience with three types of studies (sample survey, experiment, and observational study) is important at the high school level. Further, high school students should understand and appreciate the central role that randomization plays in studies and its importance in statistical inference. The random selection of subjects from a population in a sample survey tends to result in representative samples of the population. In experiments, the intent of randomly assigning treatments is to balance or equalize the treatment groups with respect to potential confounding variables that might affect the response being measured. By balancing the comparison groups, any differences measured can be attributed to the treatment effect of the explanatory variable. For an observational study with no random assignment of treatments, causality cannot be inferred. Researchers can only observe the association of response outcomes and the effect of the treatment. For an observational study where subjects are not randomly selected from a population, the scope of inference is limited; the researchers cannot generalize from the nonrandom sample to a larger population.

Students should be aware that bias might occur in several different forms: sampling bias, nonresponse bias, and response bias. It is also important for students to understand the process of quantifying the expected variability of a statistic, assuming chance variation. This can be accomplished by using the structure of sampling distributions. This important technique supports valid conclusions about population parameters. With this sampling distribution, the observed statistic from a study is either plausible under a random chance model or surprising. A measurement called the margin of error can be used to quantify sampling variability observed in a sampling distribution. The observed statistic (such as a sample proportion or sample mean), plus or minus the margin of error, provides a range of plausible values for a population parameter. All high school students should have experience using informal techniques to establish the structure of a sampling distribution that allows inferences to be made. Simulations (both hands-on and with technology) serve as powerful tools for building this understanding. The essential concepts listed below for statistical inference describe foundational knowledge necessary for carrying out the statistical investigative process.

Techniques such as simulation can help students create a visual image of the key features of a sampling distribution for a sample statistic. For example, figure 3 depicts results from two different simulations of the proportion of females in 30 randomly chosen people from a population that is known to be 50% female. Both simulated sampling distributions of the number of females are relatively symmetric around 15 females, and the number of females in the different samples of size 30 varies from about 7 to 22 or 23. Repeated simulations for the proportion of females in 30 people randomly selected from a population where the proportion of females is 0.5 will produce simulated sampling distributions with approximately the same shape.

Simulations such as these provide students with understanding of unlikely events. For example, if a random sample of 30 people under the same “50% female” assumption resulted in an observed statistic of 23 females, this result would be considered surprising as it rarely occurs by chance—an outcome that would be considered unlikely, given the 50% assumption.

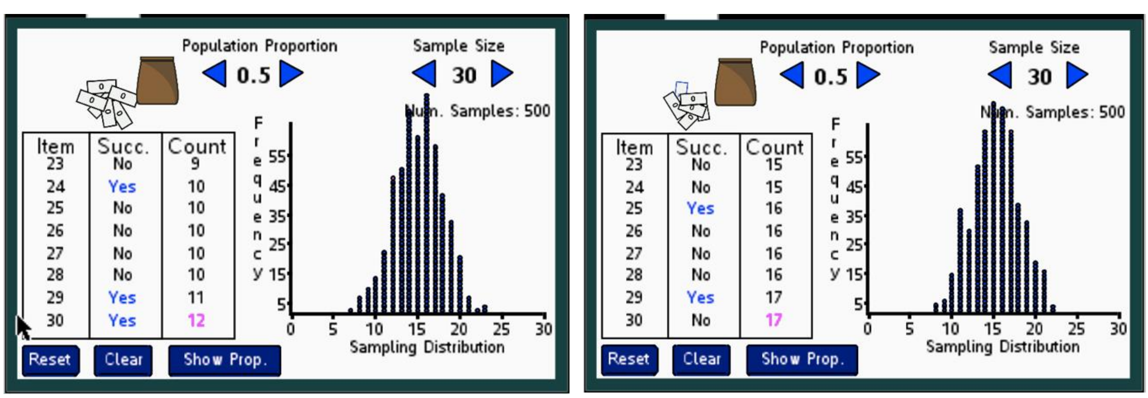


Fig. 3. Two simulations of sample size 30 for a population that is 50% female

Essential Concepts of Statistical Inference for Focus in a Common Pathway:

- **Study designs are of three main types: sample survey, experiment, and observational study.**
- **The role of randomization differs for randomly selecting samples versus randomly assigning subjects to experimental treatment groups.**
- **The scope and validity of statistical inferences are dependent on the role of randomization in the study design.**
- **Bias can occur in sample surveys if the way in which the data were gathered results in certain sample outcomes occurring in smaller or higher percentages than they occur in the population of interest.**
- **The larger the sample size, the less the variability that is expected in the sampling distribution of a sample statistic.**
- **Simulation of sampling distributions by hand or by technology can be used to determine whether a statistic (or statistical difference) is significant—whether, in a statistical sense, it is surprising or unlikely to happen when presuming that outcomes are occurring by random chance.**

Section 3: Probability

Studying and understanding probability provide high school students with a mechanism for dealing with the chance events that they experience as part of life. Probability theory brings the power of mathematics to describe randomness and chance. It helps students learn that chance variation rather than determined causation explains many phenomena. Experience with and knowledge about probability challenges intuitive ideas about chance and enables students to become members of society who have the capacity to make reasoned decisions about uncertain outcomes.

At the high school level, probability extends students' experiences with empirically and theoretically derived probabilities of chance outcomes to focus on independence of events and conditional probability. These have numerous applications in human decision making, justifying their study by all high school learners. At the same time, the caution offered by David Moore (1990, p. 122) should be considered:

> Do not dwell on combinatorial methods for calculating probabilities in finite sample spaces. Combinatorics is a different—and harder—subject than probability. Students at all levels find combinatorial problems confusing and difficult. The study of combinatorics does not advance a conceptual understanding of chance and yields less return than other topics in developing the ability to use probability modeling. …. A more fruitful step

forward from the basics of probability is to consider conditional probability, independence, and multiplication rules.

The concept of probability is addressed prior to high school, where students have experiences with theoretical and experimental probabilities. In high school, students focus on probability as emerging experimentally through a relative frequency distribution. They are able to contrast the deduced probability based on assumptions to the induced probability based on frequencies of outcome. At the high school level, all students should also develop understanding of independent events and conditional probabilities. Misunderstanding conditional probability leads to confusion between questions like “What is the probability of event A, given event B?” and “What is the probability of event B, given event A?” These two probabilities are usually different but are often confused, and the misunderstanding can have grave consequences in matters of policy, law, and other arenas.

All high school students should encounter chance events in real contexts, including situations involving both dependent and independent events, and they should understand the relationship between independence and probability. Traditionally, conditional probability has been introduced mathematically by using formulas requiring set theory concepts of intersection or complements. However, the study of conditional probability and independence of events by using contingency tables provides a context for conditional probability that is more likely to be accessible for all high school learners. This work builds on the third essential concept related to data analysis. A contingency table is a two-way table consisting of frequency counts or relative frequencies that are useful for examining associations between categorical variables. Computing conditional probabilities by using a contingency table is simplified by limiting the focus to a particular row or column known as the marginal probability. The independence of events can also be established by comparing the conditional probability from the contingency table with the marginal probability. A conditional probability can be derived by using data from a table and the summed total for the condition.

For example, the data in table 1 summarizes a survey of high school students who indicated their gender and whether or not they had a pierced ear (one or both).

Table 1. Considering gender and pierced ears

	Pierced	Not Pierced	**Total**
Female	*576*	*64*	***640***
Male	72	288	**360**
Total	**648**	**352**	**1000**

Using the data in table 1, students can examine the association between gender and ear piercing. They can understand that, for this population, the variables are not independent (both males and females can have their ears pierced), and they can compute the percentages to investigate any association between the two variables.

Figure 4 shows a graphical display of the data. On the basis of this survey data, for example, the probability that a randomly selected student has an ear pierced is 648/1000, or .648, a marginal probability that can be read from the table. For the condition “having a pierced ear,” the probability that the student will be male is 72/648, or .111. If the condition is that the student is male, then the probability that he has a pierced ear is 72/360, or .200. Each of the last two sentences is an example of conditional probability, which is computed easily from the table.

To address whether a possible association between gender and ear piecing exists, consider the marginal probability of a randomly selected student having a pierced ear and the conditional probability of a randomly selected students having a pierced ear if the student is male. Since .648 is not equal to .200, the two events are not independent, and an association exists.

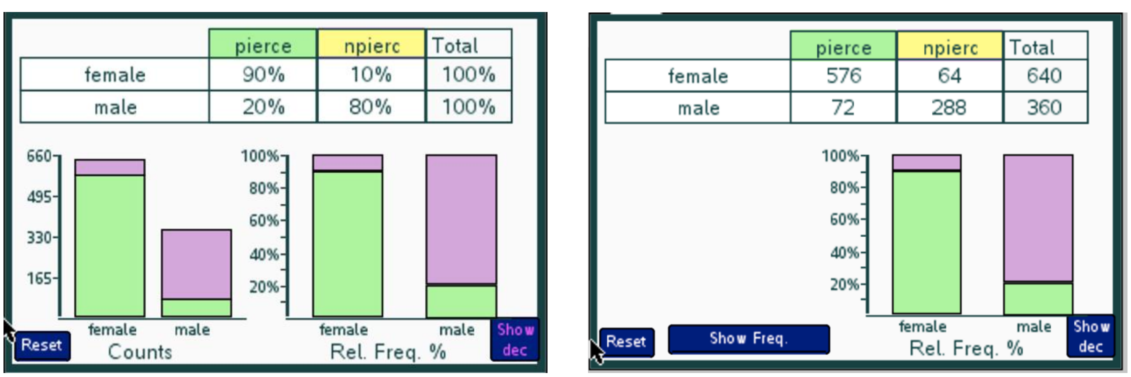


Fig. 4. Gender and pierced ears—percentages and counts in tables and bar graphs

Essential Concepts of Probability for Focus in a Common Pathway:

- **Two events are independent if the occurrence of one event does not affect the probability of the other event. Determining whether two events are independent is important for finding and understanding probabilities.**

- **Conditional probabilities—those "conditioned" by some known information—can be computed from data organized in contingency tables. Understanding how conditions or assumptions affect the computation of a probability is an important life skill because many events or models are based on conditions or assumptions.**

Section 4: Making Decisions and Quantitative Literacy

Data analysis, statistical inference, and probability offer people ways of coping with and making decision about changing events and uncertain outcomes that are inherent to personal and societal interactions. All high school students should be given opportunities to consider how concepts related to statistics and probability across many different contexts might affect their lives and should make sense of these experiences by reasoning and using their knowledge of mathematics and statistics to make informed decisions that will enable them to act accordingly.

Because members of our society must make many decisions involving data, being able to reason statistically is critically important for all high school graduates. Quantitative literacy refers to using mathematics and statistical reasoning to gain understanding in five dimensions (Steen 1997): practical (routine tasks encountered in daily life), civic (public policy matters), professional (employment skills), recreational (games, sports, and lotteries), and cultural (elements of human civilization).

Decision making too often is considered an art that relies on subjective human judgment and not a science resulting from objective, analytical, and data-driven processes. Whether the decisions are tied to health care, politics, or their personal lives, students completing high school should be able to apply the mathematical and statistical skills and knowledge that they develop in high school to inform and make decisions that confront them as members of society. Students arrive in high school with knowledge of probability and skills tied to finding probabilities of events that they have developed in earlier grades. Linking the concept of probability to weighted measures of probable gains or losses over long or repeated incidents can inform decision making by quantifying risks and benefits.

Similarly, probability and quantitative reasoning are useful in political decision making. Understanding polling results and thinking critically about the assumption underlying different polls help members of society consider how to vote or take part in democratic processes. With members of our society using social media to gain information and with social media purposefully influencing our opinions, our need to develop students who are critical thinkers, capable of sound mathematical and statistical reasoning, is great.

The concept of long-run average value is particularly important because quantifying what might be expected over the long run is accomplished by multiplying the probability times the value (typically a financial gain). This concept enables all learners to use statistical reasoning to evaluate decisions and assess relative risks, an important component of their lives as adults.

Essential Concepts of Quantitative Literacy for Focus in a Common Pathway:

- **Quantitative literacy is central to being able to evaluate conclusions and inform decision making.**

- **Reasoning based on statistical knowledge and skills and methods from data science can be used to evaluate reports and assess risks on the basis of statistical studies.**

Essential Concepts for Focus in Geometry

The English word *geometry* derives from Greek, *γεωμετρία*, with “*geo*,” or *γεω*, meaning “the earth” and “*metron*,” or *μετρία*, meaning “measurement.” Geometry provides systems, knowledge, skills, and techniques to understand, represent, and solve problems related to the space in which we live. The study of geometry—including using transformations, building understanding of measurement, using arguments and proof to develop and communicate reasoning, solving problems and using mathematics to model phenomena—develops important skills for life, postsecondary education, and careers.

High school geometry serves a special role, enabling students to learn to systematize the way that they think about shapes and motion. Whether one is trying to move a dresser around a corner in a flight of stairs, place a ladder safely, or describe the motion of an airplane, one draws on geometric ideas. By classifying types of rigid motions and understanding the results of combining them, students learn that something that may initially seem complicated—the set of all possible ways to move objects while preserving distance—can actually be decomposed into four distinct types that can be more easily grasped. Moreover, the idea of decomposing transformations into types is echoed throughout mathematics. In linear algebra and differential equations, decomposing transformations is the key to developing algorithms and finding general solutions. Students learn to look at transformations not only in terms of the result of movement but also in terms of identifying important attributes that vary or remain unchanged.

Measurement—quantifying size and shape—is a basic tool for understanding and appreciating the world that we inhabit. In ancient times, geometry was developed to assess the circumference of the earth and navigate the seas. In contemporary times, it helps explain the roundness of soap bubbles, the "page rank" of a webpage, measures of "compact areas" in legal cases on gerrymandering, and the "metrics" by which productivity is measured in business. These ideas all have foundations in geometry. Moreover, any measurement also involves making explicit underlying idealizations or contextual assumptions (e.g., the earth is a sphere, the importance of a webpage depends on how many other webpages are linked to it). It is not possible to divorce the quality of measures of a compact area in a gerrymandering case from the culture and history of the region and its geometric attributes, such as rivers or coastal lines. Soap bubbles are spherical because spheres optimize volume for a given area, and this explanation relies on the unstated assumption that nature optimizes. Clarity about principles and assumptions is essential to effective argumentation and mathematical modeling as well as for understanding others' arguments.

Geometry provides opportunities for creative investigations that open avenues both to argumentation leading to proof as well as to mathematical modeling. When constructing figures by using dynamic geometry software—for example, to examine the shapes formed by midpoints of the sides of quadrilaterals or to determine the concurrence of bisectors of various polygons—students have the opportunity to reflect on the independence of the vertices of the quadrilateral or polygons, and of the dependence on midpoints or bisectors on the vertices and edges created. From such investigations, informal arguments or formal proofs emerge. The practice of identifying independent and dependent parameters is useful not only in mathematics but also in solving problems of real life and in the sciences, humanities, and social sciences. Both geometric argumentation and mathematical modeling with geometry are essential for high school study.

Section 1: Transformations for the Study of Congruence, Similarity, and Symmetry

Transformations come in various types and are distinguished by the geometric attributes that they preserve. Transformations include rigid motions, scaling transformations, horizontal and vertical stretches, and shear transformations—among others. Equivalences between mathematical objects—such as congruence and similarity—can be formulated based on transformation types, as can symmetries of mathematical objects.

Transformations, explained by Felix Klein (1872) as part of what was known as the Erlangen Programme, greatly influenced the future of mathematics over the past century. As Douglas and Picciotto (2017) argue today, there are both pedagogical and mathematical reasons for studying geometry from a transformation viewpoint.

Pedagogically, traditional congruence postulates are not intuitive, whereas congruence as defined by means of rigid motions is very close to intuitive notions of congruence ("if you can superimpose the figures, they are congruent"). Moreover, approaching geometry from a transformation viewpoint facilitates integration with technology, such as graphing and dynamic geometry software. These technological tools allow students to work with mathematical constraints in rigorous ways and more easily share and generalize their work on tasks such as polygon construction and locus exploration.

Mathematically, transformations offer deep links between algebra and geometry, both through the concept of a transformation as a function and through their use in algebra in discussing graph transformations. Transformations are functions, and discussing the inputs and outputs of transformations, as well as using function notation for transformations, provides opportunities to discuss algebraic ideas such as inverse and composition, as well as fixed points. Such ideas play a prominent role in college-level mathematics. Graph transformations in algebra can be classified and analyzed as compositions of translations and stretch transformations. Crucially, transformations make it possible to discuss congruence or similarity of curves such as ellipses, parabolas, or other graphs that may arise in using algebra to describe natural phenomena, in ways that are far easier for students to grasp than through traditional notions of congruence that rely more heavily on triangles.

Systematizing our understanding of transformations includes classifying them, and classification is useful because it provides strategies for working with symmetry and equivalence. For instance, showing that two figures are congruent means showing that one figure can be mapped to the other by rigid motions—that is, motions that preserve length and angle measure. Showing that a figure is symmetric means showing that it can be mapped to itself by a rigid motion. Congruence of line segments of equal length, and criteria for triangle congruence and similarity (e.g., SAS, SSS, ASA) and non-criteria (e.g., SSA) provide opportunities to make congruence and similarity arguments. (It is worth noting as an aside that, in traditional Euclidean geometry, the SAS congruence criterion and the congruence of equal-length line segments are both taken as postulates. However, from a transformation perspective, where the congruence of two objects is defined as the existence of a rigid motion that maps one to the other, and reflection, rotation, and translation are postulated as rigid motions, the SAS criterion and congruence of equal-length line segments are statements that require proof and can be proven.)

Because all nontrivial rigid motions can be classified into translations, rotations, reflections, and glide reflections, establishing congruence or symmetry means finding one of these four motions. This classification is worthwhile because “finding a translation, rotation, reflection, or glide reflection” is conceptually easier than “finding a transformation that preserves length and angle measure.” Without this classification, the only information known about rigid motions is that they preserve length and angle measure. Analogously, showing that two figures are similar means showing that one figure can be mapped to the other by a scaling transformation. Scaling transformations—those motions that preserve angle measure and ratio of lengths—can be classified into dilations and their compositions with rigid motions. Shear transformations preserve area and can be used to generate different triangles with the same area, or different parallelograms with equivalent area.

The classification and definitions of rigid motion and scaling transformations afford another technique for working with congruence and similarity. The composition of a finite sequence of rigid motions is itself a rigid motion because length and angle measure are preserved by each component of the composition; and the composition of scaling transformations is itself a scaling transformation because angle measures and ratios of length are preserved by each component of the composition. Hence, *showing that two figures are congruent can consist of determining a sequence of rigid motions that maps one figure to the other, and showing that two figures are similar can consist of determining a sequence of scaling transformations that maps one figure to the other.* Also important is the foundational theorem that states that if k is the ratio of similarity between two figures, the corresponding distances are in the ratio k, the corresponding areas are in the ratio k^2, and the corresponding volumes are in the ratio k^3.

As an example of congruence arguments as an Essential Concept for Focus, consider a learning activity addressing the question of when two triangles are congruent from the Mathematics Assessment Resources Services (Mathematics Assessment Project 2017, pp. S-3, S-4). The task in this activity is framed in the following manner and uses a card set as depicted in figure 4:

Two sides of Triangle A are the same length as two sides of Triangle B. One angle in Triangle A is the same measure as an angle in Triangle B.

Draw examples of pairs of Triangles A and B that have the properties stated in the cell. Decide whether the two triangles must be congruent, and record your decision. If you decide that the triangles do not have to be congruent, draw examples and explain why. If you decide that the triangles must be congruent, try to write a convincing proof. (Mathematics Assessment Project 2017, pp. S-3, S-4)

Card Set: Must the Two Triangles be Congruent?

1. One side of Triangle A is the same length as one side of Triangle B.	2. Two sides of Triangle A are the same lengths as two sides of Triangle B.	3. Three sides of Triangle A are the same lengths as three sides of Triangle B.
4. One side of Triangle A is the same length as one side of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	5. Two sides of Triangle A are the same lengths as two sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	6. Three sides of Triangle A are the same lengths as three sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.
7. One side of Triangle A is the same length as one side of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	8. Two sides of Triangle A are the same lengths as two sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	9. Three sides of Triangle A are the same lengths as three sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.

Fig. 4. Activity cards for Must the Two Triangles Be Congruent? (Mathematics Assessment Project [2017, p. S-3])

There are two stages to this task. The first stage is experimenting with the angle and sides named in the property to understand that the property could be interpreted as the

SAS criterion or SSA or something else, depending on the location of the angle. After determining that SSA does not imply congruence, we would be ready to move to the second stage, which is experimenting with rigid motions to prove that the SAS criterion does imply congruence of triangles. For instance, we might first translate Triangle A to Triangle B so that the vertex of the named angle coincides. We then might need to reflect Triangle A about a line through the vertex so that the sides of same length are in the same orientation relative to each other as those in Triangle B. Then we can rotate one side so that it is coincident with the other side. Since rotations preserve angle measure, and we are given that the angles between the sides are the same measure, the other side must coincide. Since rotations also preserve length, the endpoints of the triangle must be coincident. Hence, the three vertices of the triangle are now coincident. Since every pair of points determines a unique line segment, we have found a sequence of rigid motions that maps Triangle A to Triangle B. Hence, if Triangle A and Triangle B satisfy the ASA criterion, they are congruent.

Showing that a figure is symmetric can consist of determining a sequence of rigid motions that maps that figure to itself. Graphs of functions and other curves provide opportunities to make congruence and similarity arguments as well as connect algebra and geometry.

Essential Concepts of Transformations for Focus in a Common Pathway:

- **Transforming different shapes, including graphs of functions, provides opportunities for describing the attributes of the shapes preserved by the transformation and for describing symmetries by examining how a shape or subset of a shape can be mapped to itself.**

- **Transformations in geometry serve as a connection to algebra through the concept of functions.**

- **Figures, including graphs of functions, are defined by their attributes, and showing that two figures are *congruent* involves finding a *rigid motion* (translation, rotation, reflection, or glide reflection) or *sequence of rigid motions* that maps one figure to the other.**

- **Figures, including graphs of functions, are defined by their attributes, and showing that two figures are *similar* involves finding a *scaling transformation* (dilation or composition of dilation with a rigid motion) or *sequence of scaling transformations* that maps one figure to the other**.

Section 2: Measurement in Two and Three Dimensions

Geometry helps us measure unknown lengths, areas, and volumes by making use of known measurements with two- and three-dimensional representations. Measurements of lengths, areas, and volumes can be computed by considering approximations, transformations, intersections, unions, and complements of objects or their lengths, areas, or volumes. Measurements can also be computed by using attributes such as height, side lengths, areas of faces, or coordinate locations.

The length or area of a curved object can be understood as a limit of a sequence of sub-lengths or sub-areas of polygonal objects that successively approximate the curved object.

As an example of the Essential Concept for Focus on constructing objects to support an understanding of measurement, consider discovering formulas for the area and circumference of a circle. These activities can provide opportunities to discuss how constructing approximating polygons and right triangles are useful for measurement. Students may explain the formulas for the area and circumference of a circle by using regular polygons with increasing numbers of sides inscribed in the circle. The essential idea of the explanation is the use of inscribed regular polygons and the right triangles within to approximate the circles. As the number of sides increases, the polygons approximate the circle, and so the area and perimeter of the polygons should approximate the circle’s area and circumference, respectively. Combining this idea with facts about triangles, the process of finding the formulas for area and circumference of a circle can foster understanding of the formulas. The approximating shapes can be thought of as a union of line segments, triangles, or rectangles, providing a useful strategy for approximations in life as well as a conceptual connection to calculus.

Constructing and measuring lengths and areas associated with two-dimensional shapes provides opportunities to dissect and rearrange the shapes into pieces such as circles, polygons, or other known figures. Constructing and measuring volumes, surface areas, and lengths associated with three-dimensional shapes provides opportunities to dissect and rearrange the shapes into pieces such as spheres, cones, pyramids, or other known figures.

Constructing right triangles is an essential technique for measurement. The length of a line segment between two coordinate points, and therefore the distance between two coordinate points, can be found by using the Pythagorean theorem. By similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Essential Concepts of Measurement for Focus in a Common Pathway:

- **Constructing approximations of measurements with different tools, including technology, can provide insights that support the understanding of measurement.**

- **Lengths, areas, and volumes of an object can be computed by determining how the object might be constructed by using products, intersections, unions, and complements of other objects and their volumes. For example,**

the volume of a cylinder is the product of the area of a cross section and its height, and the area of a ring-shaped figure is the complement of the area of the inner circle.

- **When an object is the image of a known object under a scaling transformation, the length, area, or volume of the image can be computed by using proportional relationships.**

Section 3: Geometric Arguments, Reasoning, and Proof

Geometry helps us systematize how we describe and represent objects and their motions, as well as how we make deductions about measurements and properties of objects. The geometric arguments that are possible and the ways in which objects are represented by using geometry depend on assumptions used and structures available. Technology can be used to quickly construct and compare many examples of figures that satisfy certain assumptions. Students may use technology-constructed examples to formulate conjectures about the polygons formed by connecting midpoints of the sides of a quadrilateral consecutively or determining the concurrence of medians of polygons. In these specific cases, the “midpoint” function in dynamic geometry software allows for rigorous construction of examples, and dragging one or all the vertices to different locations generates different examples. More generally, technology allows students to discuss and examine what is independent and what is dependent (e.g., vertices are independent, but midpoints depend on the vertices) and how to take these dependencies into account when constructing geometric objects. The important proportional relationship among lengths, areas, and volumes of similar figures can be investigated and established by using technology as well as through informal and formal argumentation.

Proof has traditionally been a mainstay of communicating geometric reasoning, but proof is also important in other areas of mathematics—particularly, algebra. All students should consistently experience mathematically interesting proving tasks throughout a high school geometry class. Activities that support proving, but which are not themselves proving activities, include conjecturing activities and computation activities. Proof is sometimes formatted with a two-column approach, with one column headed “statements” and the other column headed “reasons.” Although this may sometimes be a valuable way for a student to express reasoning systematically, other approaches can and should be used. Not all proofs are best communicated by the two-column approach. Students may also, for example, write sentences (paragraph proof), or use boxes (flow proof), or they may employ other formats, or combine formats, for communicating proof. Essential for both career and postsecondary education is students’ ability to communicate reasoning by means of a constructed argument in which one step leads logically to the next, complete with justification.

Geometric transformations, models, and proofs can be conceived of both synthetically and analytically. Synthetic arguments are deductive arguments based on the Euclidean axiomatic system. Finding an analytic argument or description means finding an argument or description that uses algebra with coordinates. This means using structures such as a marked origin, designated horizontal and vertical directions, and designated unit length. These structures make it possible to define attributes such as slope. Coordinates can be used to describe shapes and their motions in terms of equations of graphs and their parameters.

As an example of the Essential Concept for Focus on proofs, consider the following two proofs of the congruence of vertical angles.

Claim. *Vertical angles are congruent.*

Proof from a transformational viewpoint:

Use rotation by 180 degrees to send each line to itself (this is possible since a straight angle measures 180 degrees).

Rotation then sends vertical angles to each other.

Rotation sends angles to angles of same measure, so the vertical angles must have the same measure. ■

Proof from synthetic viewpoint:

Let x, y, x', y' be measures of angles around a point formed by two intersecting lines, labelled counterclockwise.

The measures x and x' are measures of vertical angles.

A straight angle measures 180 degrees; hence the following is true:

$x + y = 180$ degrees

$y + x' = 180$ degrees

$x' + y = 180$ degrees

$y' + x' = 180$ degrees

Hence $x = x'$ (using the first two equations) and $y = y'$ (using the last two equations). ■

This is a standard result, typically proved from a synthetic viewpoint, and most often addressed early in a geometry course. This example is presented here because it illustrates the potential for introducing different viewpoints early in geometry.

Essential Concepts of Geometric Arguments for Focus in a Common Pathway:

- **Using technology to construct and explore figures with constraints provides an opportunity to explore the independence and dependence of assumptions and conjectures.**

- **Proofs of theorems can be made from a synthetic point of view by using a deductive approach as well as from an analytical point of view by using an algebraic approach. Both approaches can be useful, and in some cases one approach may provide a more understandable argument than the other.**

- **Symmetries of figures can be useful in proving theorems by using transformations, whether the proof uses synthetic or analytic techniques.**

Section 4: Solving Applied Problems and Modeling in Geometry

High school students must be provided with problems and tasks that allow them to apply abstract geometric concepts that they have learned or are learning to solve problems, model real-world phenomena, and inform decisions. Students' experiences in geometry should include posing and solving problems situated in familiar contexts, such as landscaping, architectural designs, or urban planning. They should experience solving problems about maps, buildings, roads, or parks.

Essential Concepts of Modeling in Geometry for Focus in a Common Pathway:

- **Applying abstract geometric concepts involving congruence, similarity, measurement, trigonometry, and other related areas to solving problems situated in real-world contexts provides a means of building understanding about concepts and experiencing the usefulness of geometry.**

- **The mathematical modeling cycle using geometric concepts and methods introduces geometric techniques, tools, and points of view that are valuable to problem solving.**

Developing Equitable Common Pathways for High School Mathematics

High school students come from very diverse backgrounds with respect to culture, experience, language, socioeconomic status, personal and professional goals, and postsecondary educational goals. A high school mathematics and statistics curriculum must be designed to give this diverse set of students a mathematics education that will not only prepare them for the next steps in their future but also give them an appreciation of what mathematics is and how it can be useful in their lives, no matter what direction they take. To maximize students' opportunity after high school, it is imperative that all students complete 4 years of high school mathematics, including a mathematics or statistics course during their last year of high school.

The preceding section describes Essential Concepts for Focus with respect to the three high school content domains of algebra and functions, statistics and probability, and geometry. These essential concepts provide the foundational knowledge and experiences required to support every high school student achieving his or her goals and interests. Curriculum built on the Essential Concepts for Focus represent an equitable *common pathway* consisting of five semesters, or 2½ years, of a student's high school mathematics experience. As students traverse the common pathway, they learn and develop their skills and understanding as a common foundation for the continued study of mathematics necessary to graduate from high school prepared for a wide variety of choices among postsecondary and career options as well as for participation in democratic society. A common set of essential concepts learned in a shared setting provides the equitable education experience that all high school students deserve.

Table 1 presents three pathways as possible ways to realize the vision of high school mathematics described in *Catalyzing Change in High School Mathematics*. These options are not offered as the only possible options but are intended to serve merely as a catalyst for discussion in high schools and school districts concerning possible effective ways to organize the curriculum with the goal of ensuring that each and every student has an opportunity to learn the essential concepts for focus outlined in this publication.

Each model contains an equitable common pathway of 5 semesters that at a minimum addresses the Essential Concepts for Focus outlined in the preceding section. Offering students a common mathematics pathway in the first years of high school is a similar approach to that used in Finland to organize the curriculum (OECD 2011). This common structure would enclose a range of implementations that a school or district could choose among to best fit its unique circumstances and resources. What *Catalyzing Change* refers to as Essential Concepts 1 and 2, Essential Concepts 3 and 4, or Essential Concept 5 could represent single high school courses or be divided into two semesters built around the Essential Concepts for Focus.

Table 1. Possible pathways

Pathway	Grade 8	Grade 9	Grade 10	Grade 11	Grade 12	Postsecondary Year 1
A	Math 8	Essential Concepts 1 and 2	Essential Concepts 3 and 4	Essential Concepts 5/ Modeling	-Statistics -Precalculus - Other electives[1]	-Calculus -Finite Math - Statistics -Quantitative Literacy
B	Math 8	Essential Concepts 1 and 2	Essential Concepts 3 and 4	Essential Concepts 5/Precalculus	-Calculus	-Calculus II or III
C	Accelerated					

[1]These electives might include quantitative literacy, discrete mathematics, financial mathematics, mathematics in the fine arts, history of mathematics.

The Essential Concepts for Focus could be organized in a sequence that more closely represents the traditional U.S. progression of courses or one that uses an integrated approach to the curriculum. For example, the essential concepts could be organized to move Geometry first (semester 1) followed by Statistics/Probability (semester 2) in grade 9. This would allow all students entering grade 9 to start at the same point, giving them access to the same mathematics and building on their experiences in the middle grades. In semesters 3 and 4 (grade 10), students would continue their study of algebra, moving from their work with coordinate geometry and linearity to quadratics and exponential equations.

For Pathways A and B, semesters 5 and 6 could be structured as a yearlong course or two distinct semesters. Both options would contain the essential concepts for focus related to functions, including linear, quadratic, and exponential functions in semester 5. In Pathway A, a yearlong course might expand on this set of functions to include modeling experiences that use these functions and other aspects of the essential concepts in a variety of contexts and applied problems.

In Pathway B, a yearlong course would contain the same coverage of the essential concepts related to linear, quadratic, and exponential functions as in Pathway A, expanded to include logarithmic and trigonometric functions and other precalculus concepts to prepare students for taking calculus in high school or at the postsecondary level. Alternatively, offering two distinct semesters would allow one semester (semester 5) that provides a common experience that covers the essential concepts and a second semester (semester 6) that would provide two different options: students could choose whether to focus directly on the knowledge of functions and other precalculus-related topics that they would need if they elected to take calculus in high school (Pathway B), or they could choose to focus on modeling and/or quantitative literacy (Pathway B) to build the foundation that they would need for further study in mathematics in the final year of high school. Which second-semester path students would take could be determined by their postsecondary or career goals. The final year (2 semesters) of high school could offer students a variety of choices that might include AP Calculus, AP Statistics, Precalculus, Quantitative Literacy, Financial Mathematics, and/or Discrete Mathematics. Innovative courses that address students' interests and future needs, such as mathematics for computer science, for technology careers, or for fine arts, should be considered. It is critical that the pathways be consistent with one another with respect to rigor, relevance, and the postsecondary opportunities that they afford students.

Policies and practices must ensure that all students have an opportunity to complete the 2½ years that represent the common pathway and provide the supports necessary for them to be successful. However, Pathway C (Accelerated) acknowledges that some students might enter this pathway at a different point and some students might move through the pathway at an accelerated pace. As outlined earlier, *Catalyzing Change* argues that acceleration through the common pathway should be an exceptional practice, reserved for those students who will benefit from acceleration because their mathematical development and depth of understanding has been demonstrated to be substantially beyond grade-level learning standards (NCTM Position Statement, October 2016). Students demonstrating this complete and deep understanding beyond grade level should be provided with the opportunity to progress through the common pathway with appropriate extension that deepens their mathematical curiosity, knowledge, and skills.

If students enter the common pathway in high school without the necessary proficiency in the K–8 mathematics curriculum, additional targeted instructional supports should be provided to ensure that they are able to access the content of the common pathway beginning in grade 9, as described earlier, in the section Supporting Student Success in a Common Pathway. The vast majority of students identified with mathematics difficulties and disabilities can be successful in the common pathway with the additional targeted instructional supports outlined in that section. The Essential Concepts for Focus can serve as the foundation for the creation of mathematical learning goals in an Individualized Education Plan (IEP) or 504 plan, with appropriate and specific accommodations and/or modifications.

The expectation in *Catalyzing Change* is that a single curriculum model to deliver the common pathway would be offered to all students in a single school setting to avoid the creation of separate and unequal tracks. In other words, an individual school would not offer both an integrated and traditional pathway. Whichever curriculum pathway model schools choose or develop, the implementation should not degenerate into a situation of de facto tracking (see the earlier section *Creating Equitable Structures*). Although schools may develop additional pathways, all students should experience the Essential Concepts for Focus described in this document before they graduate from high school.

It is critical that a pathway provide flexibility and not limit students' choices—that is, no pathway should be terminal with respect to a student's future study of mathematics. This is particularly true with respect to electives that might be offered students in their final 1½ years of high school mathematics. Elective mathematics courses should be rigorous and designed in such a way that the curriculum continues to build on the Essential Concepts for Focus developed in the equitable common pathway and that postsecondary mathematics options remain open to all students. Schools and districts should simultaneously guard against a race to calculus. They should provide students with the opportunity to study calculus when their postsecondary plans require that they complete a calculus course in high school. But schools and districts must not limit the availability of rigorous alternatives for other students. For example, providing the alternative of a modeling course during semester 6 (Pathway A) and then allowing students to pick an area of focus for year 4 (precalculus, modeling, statistics, or quantitative and financial reasoning) allows students to enter postsecondary education fully prepared for a variety of majors.

Any designed pathway must contain equally rigorous mathematics focused on student reasoning and sense making (NCTM 2009) to ensure that students are positioned to use and appreciate mathematics in their personal lives and pursue additional mathematical study in a postsecondary school if that is or becomes their goal at some point in the future. In designing pathways that incorporate the Essential Concepts for Focus, school systems must take into consideration many factors, including available curricular materials and teacher capacity, and teacher collaboration and professional development time, especially for those pathways that might involve a significant change from the school’s current structure.

Catalyzing Change: Next Steps

Catalyzing Change is designed to open a serious and sustained effort on multiple levels to engage stakeholders involved in the system of high school mathematics education to improve the learning experiences and outcomes of each and every high school student. This publication offers a number of specific recommendations that can be used to begin these conversations and a number of initial actions specific stakeholders can undertake.

Key Recommendations

- High school mathematics and statistics instruction should focus on essential concepts that all students need to learn.
- High schools should offer a common shared pathway for the first 2½ years of high school mathematics that focuses on essential concepts.
- Students should enroll in the study of mathematics or statistics during each of the 4 years of high school, including a course in their last year of high school.
- High school mathematics should discontinue the practice of tracking into qualitatively different or dead-end course pathways.

Initial Actions

High School Teachers of Mathematics and Statistics

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- High school teachers of mathematics and statistics should work collaboratively and not in isolation to hold themselves and colleagues accountable for the mathematics success of each and every student and for personal and collective professional growth.

- High school teachers of mathematics and statistics should work collaboratively in subject-based teams to ensure rigorous instruction with respect to the Essential Concepts for Focus.

- High school teachers of mathematics and statistics should work collaboratively to develop mathematics instructional support systems that are fluid and provide students with targeted additional instructional time based on formative assessments that are connected with students' regular course work.

- High school teachers of mathematics and statistics should work in collaboration with building and district leaders to develop courses that meet state standards and provide the equitable common pathway for the first 2½ years of high school.

- High school teachers of mathematics and statistics should work collaboratively to link research-informed instructional practices with equity-based instructional practices.

- High school teachers of mathematics and statistics should analyze teaching assignments to develop balanced and supportive assignments to provide high-quality, engaging learning experiences for all students.

- High school teachers of mathematics and statistics should work in collaboration with building and district leaders as well as curriculum developers to develop rigorous mathematics courses that can support a variety of postsecondary and career options for students and can serve as electives for the last 1½ years of high school.

Building and District Leaders

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- Leaders should provide professional support to evaluate systemic policies, practices, and procedures that restrict access to student success in mathematics.

- Leaders should create time and space for high school teachers of mathematics and statistics to collaborate on teaching practices and work to meet the needs of each and every student.

- Leaders should create time and space for high school teachers to work collaboratively with colleagues at the K–8 level to ensure a coherent and balanced curriculum across the K–12 spectrum.

- Leaders should create time and space for students to study mathematics and statistics in all 4 years of high school.

- Leaders should provide the human and material resources to support effective teaching and learning of high school mathematics.

- Leaders should create structures within the school schedule for balanced teaching assignments.

- Leaders should discontinue the practice of tracking students into qualitatively different or dead-end course pathways and instead provide additional instructional supports as appropriate.

- Leaders should ensure that acceleration does not result in students skipping core conceptual understanding of grade-level content.

- Leaders should provide professional development support and resources for teachers to develop and implement curriculum built on the Essential Concepts for Focus.

- Leaders should provide support and resources for high school teachers to work with state and national postsecondary education leaders to ensure strong articulation between the high school and postsecondary curriculum.

- Leaders should provide professional development support and resources for teachers to implement research-informed and equity-based instructional strategies.

- Leaders should analyze and evaluate counseling practices so that every student is supported and afforded access to rigorous mathematics instruction in every year of high school.

K–8 Teachers of Mathematics and Leaders

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- K–8 teachers of mathematics and leaders should provide high-quality mathematics instruction that prepares students to use mathematics to understand and critique the world and experience the wonder, joy, and beauty of mathematics.

- K–8 teachers of mathematics and leaders should not create tracks limiting the opportunities of students to learn and experience mathematics.

Policymakers

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- Policymakers should develop policies that support an equitable common pathway that focuses students on essential concepts for the first 2½ years of high school mathematics.

- Policymakers should develop policies that support the study of mathematics or statistics for students in all four years of high school.

- Policymakers should deemphasize the “race to calculus” and increase the emphasis on developing and using the Essential Concepts for Focus.

Curriculum Developers

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- Curriculum developers should develop curriculum focused on the Essential Concepts for Focus that all students need to learn.

- Curriculum developers should offer curricula that address the equitable common pathway for the first 2½ years of high school mathematics as well as curricular options that extend beyond the common pathway.

- Curriculum developers should provide curricular materials that are grounded in research and support equitable mathematics teaching practices

Mathematics Educators

To initiate the improvement of the learning experiences and outcomes of each and every high school student—

- Mathematics educators should work with in-service teachers to support teaching focused on the Essential Concepts for Focus.
- Mathematics educators should collaborate with all stakeholders to provide resources that support the common shared pathway for the first 2½ years of high school mathematics.
- Mathematics educators should prepare preservice teachers and collaborate with in-service teachers on research-informed teaching practices that support equitable mathematics teaching practices.
- Mathematics educators should give teachers a deep understanding of the impact that education has on students' identity and levels of agency and the implications for students' futures.
- Mathematics educators should develop teachers who recognize practices and structures that disadvantage communities of students as well as strategies that address these practices and structures constructively.

The recommendations and actions outlined in *Catalyzing Change in High School Mathematics* are intended to initiate serious conversations among all stakeholders to examine how high school mathematics teaching and learning can be improved. This publication serves as a catalysis by which discussions about policies, teaching practices, pathways, and other supportive structures can be negotiated, implemented, and supported. It is hoped that this document creates a synergy that catalyzes the change and improvement that are needed to ensure that high school mathematics and statistics provide the enduring understanding, skill, interest, and enjoyment that each and every student should carry forward into postsecondary education, career, or everyday life in the 21st century.

References

Achieve. (2013). *Closing the expectations gap: 2013 annual report on the alignment of state K-12 policies and practice with the demands of college and careers.* Washington, DC: Author.

ACT. (2016). *The condition of college and career readiness 2016: National.* Iowa City, IA: Author.

AERA. (2006). Do the math: Cognitive demand makes a difference. *Research Points: Essential Information for Education Policy*, 4(2).

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). *The impact of identity in K–8 mathematics: Rethinking equity-based practices*. National Council of Teachers of Mathematics: Reston, VA.

Baker, S., Gersten, R., & Lee. D. (2002). A synthesis of empirical research on teaching mathematics to low-achieving students. *The Elementary School Journal,* 103(1), 51-73.

Balfanz, R., MacIver, D. J., & Byrnes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high poverty middle schools: Multi-school, multi-year studies. *Journal for Research in Mathematics Education,* 37, 33-64.

Barber, M., & Mourshed, M. (2007). *How the world's best-performing school systems come out on top.* McKinsey & Company.

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the Standards for Mathematical Practice. *Journal for Research in Mathematics Education*, *48*(1), 7-21.

Bass, H. (2015). Mathematic and Teaching. *I, Mathematician*. Mathematical Association of America

Berry III, R. Q. (2016). Informing teachers about identities and agency: Using the stories of Black middle school boys who are successful with school mathematics. In E. Silver & P. A. Kenney (Eds.), *More lessons learned from research: Helping all students understand the important mathematics; Volume 2* (pp. 25—37). Reston, VA: National Council of Teachers of Mathematics.

Berry III, R. Q. & Berry, M. P. (2017). Professionalism: Building a Culture by Creating Time and Space. In D. Spangler & J. Wanko (Eds.) *Enhancing classroom practice with research behind Principles to Actions.* Reston, VA: National Council of Teachers of Mathematics.

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning.* Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Boaler, J. (2011). Changing students' lives through the de-tracking of urban mathematics classrooms. *Journal of Urban Mathematics Education*, 4(1), 7-14.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching.* San Francisco, CA: Jossey-Bass.

Boaler, J., & Staples, M. E. (2014). Creating mathematical futures through an equitable teaching approach: The case of Railside school. In N. S. Nasir, C. Cabana, B. Shreve, E. Woodbury, & N. Louie (Eds.), *Mathematics for equity: A framework for successful practice* (pp. 11-34). New York: Teachers College Press.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608–645.

Boaler, J., Wiliam, D., & Brown, M. (2000). Students' experiences of ability grouping—disaffection, polarisation and the construction of failure. *British Educational Research Journa*l, 26(5), 631-648.

Boston, M. Dillon, F. Smith, M. S., & Miller, S. (2017). *Taking Action: Implementing Effective Mathematics Teaching Practices Grades 9-12.* NCTM: Reston VA.

Boston, M., & Smith, M. (2009). Transforming Secondary Mathematics Teaching: Increasing the Cognitive Demands of Instructional Tasks Used in Teachers' Classrooms. *Journal for Research in Mathematics Education, 40*(2), 119-156. Retrieved from http://www.jstor.org/stable/40539329

Bressoud, D. (2015). Insights from the MAA national study of college calculus. *Mathematics Teacher*, 109(3), 179-185.

Bressoud, D., Mesa, V., & Rasmussen, C. (2015). *Insights and recommendations from the MAA national study of college calculus.* Washington, D.C.: Mathematical Association of America.

Burris, C. C., Heubert, J. P., & Levin, H. M. (2006). Accelerating mathematics achievement using heterogeneous grouping. *American Educational Research Journal,* 43(1), 105-136.

Burris, C. C., & Weiner, K. G. (2005). Closing the achievement gap by detracking. *Phi Delta Kappan,* 86(8), 594-598.

Carlson, M., Madison, B., & West, R. (2010). The Calculus Concept Readiness (CCR) instrument: Assessing student readiness for calculus. arXiv preprint. Retrieved May 15, 2017 from https://arxiv.org/pdf/1010.2719.pdf.

Cobb, G., & Moore D. (1997). Mathematics, statistics, and teaching. *The American Mathematical Monthly*, 104, (9) pp. 801-823. Mathematical Association of America

Cortes, K., Goodman, J., & Nomi, T. Intensive math instruction and educational attainment: Long-run impacts of double-dose algebra. *Journal of Human Resources,* 50(1), 108-158.

Cuoco, A. , Goldenberg, E. P., & Mark, J. (2010). Contemporary curriculum issues: Organizing a curriculum around mathematical habits of mind. *Mathematics Teacher,* 103(9), 682-688.

D’Ambrosio, U. (2012). A broad concept of social justice. In A. A. Wager & D. W. Stinson (Eds.), *Teaching mathematics for social justice: Conversations with educators* (pp. 201-213). Reston, VA: NCTM.

Darling-Hammond, L. (2007). The flat earth and education: How America’s commitment to equity will determine our future. *Educational Researcher*, 36(6), 318-334.

Dossey, J. A., McCrone, S. S., & Halvorsen, K. T. (2016). *Mathematics education in the United States 2016: A capsule summary fact book.* Reston, VA: NCTM.

Douglas, L. & Picciotto, H. (2017). *Transformational Proof in High School Geometry*. Retrieved from https://www.mathedpage.org/transformations/proof/transformational-proof.pdf

Drijvers, P., Boon, P., & Van Reeuwijk, M. (2010). Algebra and technology. In P. Drijvers (Ed.), *Secondary algebra education. Revisiting topics and themes and exploring the unknown* (pp. 179–202). Rotterdam: Sense.

Education Trust. (2005). *Gaining traction, gaining ground: How school schools accelerate learning for struggling students.* Washington, DC: Author.

Ernest, P. (2010). Why teach mathematics. *Professional Educator*, 9(2), 44-47.
Gutiérrez, R. (2012). Embracing Nepantla: Rethinking “knowledge” and its use in mathematics teaching. *REDIMAT—Journal of Research in Mathematics Education*, 1(1), 29-56.

Feiman-Nemser, S. (2017). Beyond solo teaching. *Educational Leadership 69*(8), 10-16.

Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *High School Journal*, 91, 29-42.

Flores, A. (2008). The opportunity gap. *TODOS research monograph. Promoting high participation and success in mathematics by Hispanic students: Examining opportunities and probing promising practices,* 1(1), 1-18.

Franklin, C., Bargagliotti, A., Case, C., Kader, G., Schaeffer, R., & Spangler, D., (2015). *The Statistical Education of Teachers.* Arlington, VA: American Statistical Association.

Franklin, C., Kador, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., Scheaffer, R. (2005). *Guidelines for assessment and instruction in statistics education (GAISE) report: A Pre-K–12 curriculum framework*. Alexandria VA: American Statistical Association.

GAISE College Report ASA Revision Committee, "Guidelines for Assessment and Instruction in Statistics Education College Report 2016," http://www.amstat.org/education/gaise .

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing Competence: An Analysis of Student Participation in the Activity Systems of Mathematics Classrooms. *Educational Studies in Mathematics, 70*(1), 49-70.

Grissom, J. A., & Redding, C. (2016). Discretion and disproportionality: Explaining the underrepresentation of high-achieving students of color in gifted program. *AERA Open,* 2(1), 1-25.

Gutiérrez, R. (1996). Practices, beliefs, and cultures of high school mathematics departments: Understanding their influences on student advancement. *Journal of Curriculum Studies, 28*(5) 495–530.

Gutiérrez, R. (2002a). Enabling the practice of mathematics teaches in context: Toward a new equity research agenda. *Mathematical Thinking and Learning,* 4(2&3), 145-187.

Gutiérrez, R. (2002b). Beyond essentialism: The complexity of language in teaching mathematics to Latina/o students. *American Educational Research Journal,* (39), 1047-88.

Gutiérrez, R. (2012). Embracing Nepantla: Rethinking "knowledge" and its use in mathematics teaching. *REDIMAT—Journal of Research in Mathematics Education*, 1(1), 29-56.

Gutstein, E. (2016). Our issues, our people—math as our weapon: Critical mathematics in a Chicago neighborhood high school. *Journal for Research in Mathematics Education,* 47(5), 454-504.

Gutstein, E., & Peterson, B. (2013). Introduction. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers, second edition* (pp. xi-xiii). Milwaukee, WI: Rethinking Schools.

Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. *Educational Studies in Mathematics*, *80*(1-2), 233-247.

Heitin, L. (2015, February 24). Common core seen falling short in high school math. *Education Week.* http://www.edweek.org/ew/articles/2015/02/25/common-core-seen-falling-short-in-high.html

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr., (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age Publishing.

Horn, I. S. (2005). Learning on the job: A situated account of teacher learning in high school mathematics departments. *Cognition and Instruction*, *23*(2), 207-236.

Horn, I. S. (2006). Lessons learned from detracked mathematics departments. *Theory into Practice*, 45(1), 72-81.

Isenberg, E., Max, J., Gleason, P., Potamites, L., Santillano, R., Hock, H., & Hansen, M. (2013). *Access to effective teaching for disadvantaged students* (NCEE 2014-4001). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Kanold, T. D., & Larson, M. R. (2015). *Beyond the common core: A handbook for mathematics in a PLC at Work™*. Reston, VA: NCTM and Solution Tree Press.

Klein, F. (1872). A Comparative Review of Recent Researches in Geometry. Translated by M.W. Haskell, *Bulletin of the New York Mathematical Society, 2* (1892-1893), pp. 215-249. Retrieved from http://math.ucr.edu/home/baez/erlangen/erlangen_tex.pdf

Larnell, G. V., Bullock, E. C., & Jett, C. C. (2016). Rethinking Teaching and Learning Mathematics for Social Justice from a Critical Race Perspective. *Journal of Education*, *196*(1).

Larson, M. R., & Andrews, D. (2015). One district's journey to promote access and equity. *New England Mathematics Journal,* 47, 31-40.

Larson, M. R. (2016). *Bringing needed coherence and focus to high school mathematics.* Presidential Message (October 25, 2016). http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Matt-Larson/Bringing-Needed-Coherence-and-Focus-to-High-School-Mathematics/

Leinwand, S. (2015). Professionalism: The Newest Principle. *New England Mathematics Journal, 46*(1), 53-63.

Levasseur, K. & Cuoco, A. (2003). Mathematical habits of mind. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6-12* (pp. 23-37). Reston, VA: NCTM.

Loveless, T. (2013). *The 2013 Brown Center report on American education: How well are American students learning?* Washington, DC: Brown Center on Education Policy, Brookings Institution.

Lubienski, S. T. (2007). What can we do about achievement disparities? *Educational Leadership*, 65(3), 54-59.

Martin, D. B. (2015). The collective black and Principles to Action. *Journal of Urban Mathematics Education*, *8*(1), 17-23.

Mathematics Assessment Project. (2017). *Evaluating conditions for congruency.* Nottingham, UK, & Berkeley, CA: Mathematics Assessment Resource Service (MARS), University of Nottingham & UC Berkeley. Retrieved from http://map.mathshell.org/download.php?fileid=1752

Moore, D. (1990). Uncertainty. In L. A. Steen (Ed.) *On the Shoulder of Giants: New Approaches to Numeracy*. Washington, D.C.: National Academy Press, 1990

Moore, D. (1997). Probability and statistics in the core curriculum. In J. Dossey (ed.), *Confronting the Core Curriculum*, Mathematical Association of America, 93(98).

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS Advanced 2015 International Results in Advanced Mathematics and Physics.* Retrieved from Boston College, TIMSS & PIRLS International Study Center website: http://timssandpirls.bc.edu/timss2015/international-results/advanced/

Murrell, P.C. (2007). Race, culture, and schooling: Identities of achievement in multicultural urban schools. New York, NY: Erlbaum.

Nasir, N. S. (2016). Why should mathematics educators care about race and culture? *Journal of Urban Mathematics Education,* 9(1), 7-18.

Nasir, N. S., & Hand, V. M. (2006). Exploring sociocultural perspectives on race, culture, and learning. *Review of Educational Research*, *76*(4), 449-475.

National Association of Colleges and Employers (NACE). (2016). *STEM grads projected to earn class of 2016's highest average starting salaries.* Bethlehem, PA: National Association of Colleges and Employers. Downloaded from http://www.naceweb.org/job-market/compensation/stem-grads-projected-to-earn-class-of-2016s-highest-average-starting-salaries/

National Center for Education Statistics (NCES). (2015). *The nation's report card: 2015 mathematics and reading at grades 4 and 8* (NCES 2015-136). Washington, DC: Author.

National Center for Education Statistics. (2016). *2015 results mathematics & reading grade 12.* https://www.nationsreportcard.gov/reading_math_g12_2015/files/infographic_2015_g12_math_reading.pdf

National Commission on Excellence in Education. Department of Education. (1983). *A nation at risk : the imperative for educational reform : a report to the Nation and the Secretary of Education,* United States Department of Education. Washington, D.C. :The Commission : [Supt. of Docs., U.S. G.P.O. distributor]

National Council of Teachers of Mathematics. (1980). *An agenda for action*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2009). *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

NCTM. (2016). *Providing opportunities for students with exceptional promise.* Reston, VA: NCTM. http://www.nctm.org/Standards-and-Positions/Position-Statements/Providing-Opportunities-for-Students-with-Exceptional-Promise/

NCTM & MAA. (2012). *Joint position statement on calculus.* Reston, VA. http://www.nctm.org/Standards-and-Positions/Position-Statements/Calculus/

National Research Council [NRC]. (2011). *Fueling innovation and discovery: The mathematical sciences in the 21st century.* Washington, DC: The National Academies Press.

National Research Council. (2012). Education for Life and Work: Developing Transferable Knowledge and Skills in the 21st Century. Committee on Defining Deeper

Learning and 21st Century Skills, James W. Pellegrino and Margaret L. Hilton, Editors. Board on Testing and Assessment and Board on Science Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

NGA & CCSSO. (2010). *Common Core State Standards for Mathematics*.

Oakes, J. (1985). *Keeping track: How schools structure inequality.* New Haven, CT: Yale Press.

Oakes, J. (2008). Keeping track: Structuring equality and inequality in an era of accountability. *Teachers College Record*, 110(3), 700-712.

Oakes J., Ormseth T., Bell R., Camp P. (1990). Multiplying inequalities: The effects of race, social class and tracking on opportunities to learn mathematics and science. Santa Monica, CA: RAND.

OECD. (2011). *Lessons from PISA for the United States, Strong Performers and Successful Reformers in Education*. OECD Publishing. http://dx.doi.org/10.1787/9789264096660-en

OECD. (2016a). *Equations and inequalities: Making mathematics accessible to all.* Paris: PISA OECD Publishing. Downloaded at http://dx.doi.org/10.1787/9789264258495-en.

OECD. (2016b). *PISA 2015 Results (Volume I): Excellence and Equity in Education*, OECD Publishing, Paris.

Oppland-Cordell, S., & Martin, D. B. (2015). Identity, power, and shifting participation in a mathematics workshop: Latin@ students' negotiation of self and success. *Mathematics Education Research Journal*, *27*(1), 21-49.

Pollak, H. (2012). What is mathematical modeling? In H. Gould, D. Murray, & A. Sanfratello (Eds.), *Mathematical Modeling Handbook* (Introduction, pp. viii-xi). Bedford, MA: COMAP.

President's Council of Advisors on Science and Technology (PCAST). (2012). *Report to the president, engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, DC: Executive office of the president, president's council of advisors on science and technology. Downloaded from https://obamawhitehouse.archives.gov/sites/default/files/microsites/ostp/pcast-engage-to-excel-final_2-25-12.pdf

Roschelle, J., Pea, R. D., Hoadley, C. M., Gordin, D. G., & Means, B. (2000). Changing how and what children learn in school with computer-based technologies. *The Future of*

Children, 10(2), 76–101.

Sacristán, A. I., Calder, N., Rojano, T., Santos, M., Friedlander, A., & Meissner, H. (2010). The influence and shaping of digital technologies on the learning—and learning trajectories—of mathematical concepts. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain*. The 17th ICMI study (Vol. 13, New ICMI Study Series, pp. 179–226). New York, NY: Springer.

Saxe, K., & Braddy, L. (2015). A common vision for undergraduate mathematical sciences programs in 2025. Washington DC: Mathematical Association of America.

Schmidt W. H. (2009). Exploring the relationship between content coverage and achievement: Unpacking the meaning of tracking in eighth grade mathematics. East Lansing: Michigan State University, Education Policy Center. Retrieved from http://education.msu.edu/epc/forms/Schmidt_2009_Relationship_between_Content_Coverage_and_Achievement.pdf

Schmidt, W. H., Cogan, L. S., & McKnight, C. C. (2011). Equality of Educational Opportunity: Myth or Reality in US Schooling? *American Educator*, 34 (4), 12–19.

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education*, 10, 167–182.

Seeley, C. (2006). "Technology is a tool: President's message." NCTM News Bulletin 42(7): 3.

SIAM (2016). *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)*. Retrieved from http://www.siam.org/reports/gaimme.php

StatTrek (2017). *Two-Way Tables in Statistics*. Retrieved from http://stattrek.com/statistics/two-way-table.aspx?Tutorial=AP

Stein, M. K., Kaufman, J. H., Sherman, M., & Hillen, A. F. (2011). A challenge at the crossroads of policy and practice. *Review of Educational Research,* 81(4), 453-492.

Stiff, L. V., and Johnson, J. L. (2011). Mathematical reasoning and sense making begins with the opportunity to learn. In M. E. Strutchens and J. R. Quander (Eds.), *Focus in high school mathematics: Fostering reasoning and sense making for all students* (pp. 85-100). Reston, VA: NCTM.

Stinson, D. W., & Wager, A. A. (2012). A sojourn into the empowering uncertainties of teaching and learning mathematics for social change. In D. W. Stinson and A. A. Wager (Eds.), *Teaching mathematics for social justice: Conversations with educators* (3-18). Reston, VA: NCTM.

Steen, L. (1997). *Why Numbers Count: Quantitative Literacy for Tomorrow's America*. New York, NY: College Entrance Education Board.

Strutchens, M. E., Quander, J. R., & Gutiérrez, R. (2011). Mathematics learning communities that foster reasoning and sense making for all high school students. In M. E. Strutchens and J. R. Quander (Eds.), *Focus in high school mathematics: Fostering reasoning and sense making for all students* (pp. 101-113). Reston, VA: NCTM.

Su, F. (2017). Mathematics for Human Flourishing. *American Mathematical Monthly*, 124, 483-493.

Tate, W., & Rousseau, C. (2002). Access and opportunity: The political and social context of mathematics education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 271-300). Mahwah, NJ: Erlbaum.

Tate, W. F. (2013). Race, retrenchment, and the reform of school mathematics. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers, second edition* (pp. 42-51). Milwaukee, WI: Rethinking Schools.

Thompson, K. D. (2017). What blocks the gate? Exploring current and former English learners' math course-taking in secondary school. *American Educational Research Journal,* 54(4), 757-798.

Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positioning and dynamic identities. *Journal for Research in Mathematics Education*, *44*(1), 199–234.

Usiskin, Z. (1995). Why is Algebra Important to Learn? *American Educator,* 19(1), 30-37.

Usiskin, Z. (1988, 2014). Conceptions of Algebra and Uses of Variables. Reprinted in *We Need Another Revolution in School Mathematics: Five Decades of Curriculum Papers by Zalman Usiskin.* Reston VA: NCTM.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571-596.

Vilorio, D. (2014). *STEM101: Intro to tomorrow's jobs.* Washington DC: U.S. Department of Labor, Bureau of Labor Statistics.

Wenger, E. (1998). Communities of practice: learning, meaning and identity. Cambridge: Cambridge University Press.

Wolfe, I. (2013). *65% of today's students will be employed in jobs that don't exist yet.* Success performance solutions. Downloaded from

https://www.successperformancesolutions.com/65-percent-of-todays-students-will-be-employed-in-jobs-that-dont-exist-yet/