A Teacher's Guide to Reasoning and Sense Making

WHAT KINDS of experiences should high school mathematics offer students? The key is to provide important mathematical opportunities centered on reasoning and sense making. How can you and other high school mathematics teachers like you give students these kinds of experiences?

Focus in High School Mathematics: Reasoning and Sense Making, a new publication from the National Council of Teachers of Mathematics, offers guidelines to improve high school mathematics by refocusing it in this way. Teachers will play a crucial role in realizing the vision of this innovative publication from the nation's leading advocate for more and better mathematics.

What do *reasoning* and *sense* making mean?

Reasoning and *sense making* refer to students' abilities to think about and use mathematics in meaningful ways. In any subject, simply exposing students to topics is not enough. Nor is it enough for students to know only how to perform procedures. For example, in your own experiences with mathematics in high school, you may have solved page after page of equations or factored page after page of polynomials. Students today must understand more about algebra than how to apply procedures. They need to develop critical thinking skills to succeed in the subject—and in other areas of life and learning.

For instance, in high school literature courses, students often must analyze, interpret, or think critically about books that they are reading. Reasoning is important in all fields—particularly mathematics. Mathematical reasoning involves drawing logical conclusions on the basis of assumptions and definitions. Sense making involves developing an understanding of a situation, context, or concept by connecting it with other knowledge. Reasoning and sense making are closely interrelated.

Reasoning and sense making should occur in every mathematics classroom every day. In classrooms that encourage these activities, teachers and students ask and answer such questions as "What's going on here?" and "Why do you think that?" Addressing reasoning and sense making does not need to be an extra burden if you are working with students who are having a difficult time just in learning procedures. On the contrary, the structure that a focus on reasoning brings can provide vital support for understanding and continued learning.

Often students struggle because they find mathematics meaningless. Instruction that fails to help them find connections through reasoning and sense making may lead to a seemingly endless cycle of reteaching. However, with purposeful attention and planning, teachers can hold all students in every high school mathematics classroom accountable for personally engaging in reasoning and sense making, thus leading students to reason for themselves instead of merely observing and applying the reasoning of others.

What can you do in your classroom to ensure that reasoning and sense making are paramount?

You can make reasoning and sense making a focus in any mathematics class. A crucial step is to determine how reasoning and sense making serve as integral components of the material that you teach.

Even with topics traditionally presented through procedural approaches, you can teach the concepts in ways that allow students to reason about what they are doing. Although procedural fluency is important in high school mathematics, it should not be sought in the absence—or at the expense—of reasoning and sense making.

What exactly do reasoning and sense making "look like" in the mathematics classroom? The following example illustrates the need to infuse reasoning and sense making into a classroom experience. The scenario illustrates what frequently happens when students are asked to recall a procedure taught without understanding—in this case, the distance formula.

Teacher: Today's lesson requires that we calculate the distance between the center of a circle and a point on the circle to determine the circle's radius. Who remembers how to find the distance between two points?

Student 1: Isn't there a formula for that?

Student 2: I think it's x_1 plus x_2 squared, or something like that.

Student 1: Oh, yeah, I remember—there's a great big square root sign, but I don't remember what goes under it.

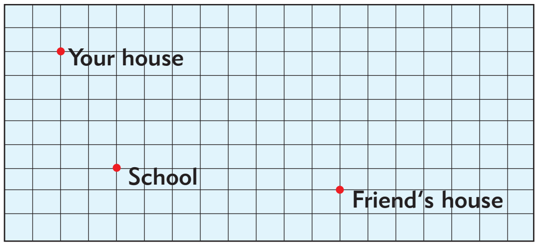
Student 3: I know! It's x_1 plus x_2, all over 2, isn't it?

Student 4: No, that's the midpoint formula.

The discussion continued in the same way until the teacher reminded the class of the formula. The next year, the same teacher decided to try a different approach—one with the potential to engage the students in reasoning about the distance formula as they solved a problem. The following scenario shows students reasoning about mathematics, connecting what they are learning with the knowledge that they already have and making sense of the distance formula:

Teacher: Let's take a look at a situation in which we need to find the distance between two locations on a map. Suppose that this map [*shown at the top of the next page*] shows your school; your house, which is located two blocks west and five blocks north of school; and your best friend's house, which is located eight blocks east and one block south of school. Also suppose that the city has a system of evenly spaced perpendicular and parallel streets. How many blocks would you have to drive to get from your house to your friend's house?

Student 1: Well, we would have to drive ten blocks to the east and six blocks to the south, so I guess it would be sixteen blocks, right?

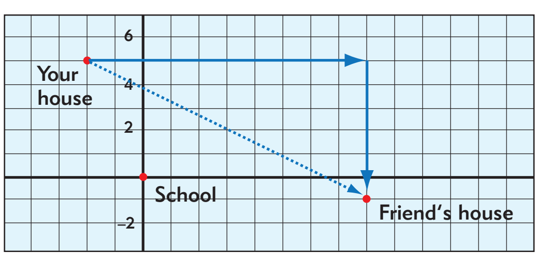


Teacher: But what if you could use a helicopter to fly straight to your friend's house? How could you find the distance "as the crow flies"? Work with partners to establish a coordinate-axis system and show the path that you'd have to drive to get to your friends house. Then work on calculating the direct distance between the houses if you could fly.

Student 1: *[working with students 2, 3, and 4]:* What if we use the school as the origin? Then wouldn't my house be at (–2, 5) and my friend's house be at (8, –1)?

Student 2: Yeah, that sounds right. Here, let's draw the path on the streets connecting the two houses, and then draw a line segment connecting the two houses.

Student 1: Maybe we could measure the length of a block and find the distance with a ruler.



Student 3: Wait a minute—you just drew a right triangle, because the streets are perpendicular.

Student 4: So that means we could use the Pythagorean theorem:
$0^2 + 6^2 = c^2$, so $c = \sqrt{136}$

Student 2: But how many blocks would that be?

Student 3: Shouldn't the distance be between eleven and twelve blocks, since $121 < 136 < 144$? Actually, it's probably closer to twelve blocks, since 136 is much closer to 144 than to 121.

The teacher then extended the discussion to consider other examples and finally to develop the general formula. By having the students approach the distance formula from the perspective of reasoning and sense making, she increased their understanding of the formula and why it is true, making it more likely that they would be able to retrieve, or quickly recreate, the formula later.

The focus of every mathematics class should be on helping students make sense of the mathematics for themselves. Bringing this focus to instruction depends on—

- selecting worthwhile tasks that engage and develop students' mathematical understanding, skills, and reasoning;
- creating a classroom environment in which serious engagement in mathematical thinking is the norm;
- effectively orchestrating purposeful discourse aimed at encouraging students to reason and make sense of what they are

- using a range of assessments to monitor and promote reasoning and sense making, both in identifying student progress and in making instructional decisions;
- constantly reflecting on teaching practice to be sure that the focus of the class in on reasoning and sense making (based on recommendations in *Mathematics Teaching Today* [NCTM 2007]).

The teacher in the preceding example performed each of these actions with the apparent goal of helping students move beyond simply knowing how to find the distance by using a formula, to understanding and making sense of the formula itself.

What should you expect students to be able to do?

Focus in High School: Reasoning and Sense Making describes reasoning habits that should become routine and fully expected in all mathematics classes at all levels of high school. Approaching these reasoning habits as new topics to be taught is not likely to have the desired effect. The crowded high school mathematics curriculum affords little room for introducing them in this way. Instead, you should give attention to reasoning habits and integrate them into the existing curriculum to ensure that your students both understand and can use what you teach them.

Reasoning habits involve—

- *Analyzing a problem,* for example—
  - — *identifying relevant* mathematical *concepts, procedures, or representations* that reveal important information about the problem and contribute to its solution (for example, choosing a model for simulating a random experiment);
  - — *defining relevant variables and conditions* carefully, including units if appropriate;
  - — *seeking patterns and relationships* (for example, systematically examining cases or creating displays for data);
  - — *looking for hidden structures* (for example, drawing auxiliary lines in geometric figures, finding equivalent forms of expressions that reveal different aspects of the problem);
  - — *considering special cases or simpler analogs;*
  - — *applying previously learned concepts* to the problem, adapting and extending as necessary*;*
  - — *making preliminary deductions and conjectures*, including predicting what a solution to a problem might look like or putting constraints on solutions*;* and
  - — *deciding whether a statistical approach is appropriate*.
- *Implementing a strategy,* for example—
  - — *making purposeful use of procedures;*
  - — *organizing the solution*, including calculations, algebraic manipulations, and data displays;
  - — *making logical deductions* based on current progress, verifying conjectures, and extending initial findings; and
  - — *monitoring progress toward a solution*, including reviewing a chosen strategy and other possible strategies generated by oneself or others.

- *Seeking and using connections* across different mathematical domains, different contexts, and different representations.
- *Reflecting on a solution to a problem,* for example—
  - — *interpreting a solution* and how it answers the problem, including making decisions under uncertain conditions;
  - — *considering the reasonableness* of a solution, including whether any numbers are reported to an unreasonable level of accuracy;
  - — *revisiting initial assumptions* about the nature of the solution, including being mindful of special cases and extraneous solutions;
  - — *justifying or validating* a solution, including proof or inferential reasoning;
  - — *recognizing the scope of inference* for a statistical solution;
  - — *reconciling different approaches* to solving the problem, including those proposed by others;
  - — *refining arguments* so that they can be effectively communicated; and
  - — *generalizing a solution* to a broader class of problems and looking for connections to other problems.

Many of these reasoning habits fit in more than one category, and students should move naturally and flexibly among them as they solve problems and think about mathematics. *Focus in High School: Reasoning and Sense Making* offers examples of ways to promote these habits in the high school classroom.

What can you do to help students understand the importance of mathematics in their lives and future career plans?

Knowing and using mathematics in meaningful ways are important for all students, regardless of their post–high school plans. Whether the students attend college and major in mathematics or go straight into the workforce after graduation, they will need to have confidence in their knowledge of and ability to use mathematics.

To help students realize the importance of mathematics in their lives, you should recognize and demonstrate the need for mathematics reasoning habits and content knowledge as essential life skills. You must show how these skills can ensure your students' success for many years to come—not just in the next mathematics course that the students may take.

In addition, you should demonstrate an awareness of the wide range of careers that involve mathematics, including finance, real estate, marketing, advertising, forensics, and even sports journalism. Exposing students to the ways in which fields such as these use mathematics will help them appreciate the importance of mathematics in their own lives.

Beyond showing the relevance of mathematics in an array of careers, you should also emphasize its practical value in offering approaches to real problems. Seek contexts in which your students can see that mathematics can be a useful and important tool for making decisions. In doing so, you will help students recognize the benefit of mathematical reasoning and its importance for their adult lives. Such lessons can contribute to the development of a productive disposition toward mathematics.

What can you do to make your students' high school mathematical experiences more meaningful overall?

You can be an important advocate beyond your own classroom for more meaningful high school mathematics. Compared with teachers of mathematics in the middle and elementary grades—or with school administrators at any level—high school mathematics teachers generally have stronger, more extensive mathematics backgrounds and have taken higher-level mathematics courses.

Because of these experiences, high school mathematics teachers are the most likely to see mathematics as a coherent subject in which the reasons that results are true are as important as the results themselves. You can play a vital role in communicating that message to other decision makers in your school.

For the experience of learning high school mathematics to change and become something that is meaningful to your students, you must begin today to focus your content and instruction on reasoning and sense making. In addition, you are in a unique position to work with administrators and policymakers to achieve the goal of broadly restructuring the high school mathematics program to reflect this focus.

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